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V. *On the Application of Harmonic Analysis to the Dynamical Theory of the Tides.—* Part II. *On the General Integration of* LAPLACE'S *Dynamical Equations.*

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IN the former paper on this subject I have dealt with the formation of LAPLACE'S dynamical equation for the tides, and the integration of it, subject to the limitation that the solutions obtained should be symmetrical with respect to the axis of rotation. In the present paper I propose to extend the method of solution so as to free it from this restriction.

The difficulties experienced by LAPLACE in his attempts to integrate the equation in question were so great that he abandoned all efforts to obtain a general solution, and confined his discussion to a few of the special cases which present the greatest interest from a practical point of view; even in these simple cases however his original attempts to express the solutions by means of the coefficients associated with his name were discarded in favour of series proceeding according to powers of a certain variable used to define the position of a point on the Earth's surface. These power-series have been further employed by Lord KELVIN* to obtain a more general solution of the problem, but the results obtained, though of considerable analytical interest, do not lend themselves well to a numerical discussion. Both AIRY† and KELVIN condemn the employment of the surface-harmonic functions as inappropriate, but a profound conviction that the efforts of LAPLACE, though unsuccessful, were well directed, has led me to take up the problem again from his point of view; with what success will be seen hereafter.

I was originally led to attack the problem by a totally different method from that of LAPLACE based on the work of POINCARÉ‡ and BRYAN§, and the principal analytical results, both in this paper and in the preceding, were at first obtained by

* "On the General Integration of LAPLACE'S Differential Equation of the Tides." 'Phil. Mag.,' 1875.

† 'Encyc. Metropolitana.' Art. "Tides and Waves," Section III., § 116.

‡ "Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation." 'Acta Math.,' vol. 7, p. 355, *et seq.*

§ "The Waves on a Rotating Liquid Spheroid of Finite Ellipticity." 'Phil. Trans.,' A, 1889.

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a very lengthy analysis similar to that used by the latter writer. The comparative simplicity of these results seemed, however, to point to the fact that they might be more easily obtained by less pretentious means. The deduction of the formulæ in the former paper from the differential equation of LAPLACE presented no serious difficulties, but in attempting to apply a like method to obtain the more general formulæ of the present paper, I found that formidable obstacles had to be overcome. The method of integration now adopted seems to leave little to be desired for simplicity, considering its generality, but the fact that it has been built up partly by working forwards from the differential equation, and partly by working backwards from the results, must account for the apparent artificiality of the procedure.

In all previous attempts at the solution of the dynamical equations for the tides, the integration has been effected by assuming that the expression for the tide-height could be expressed by an infinite series of terms of known form associated with undetermined numerical coefficients. The differential equations then lead to a difference-relation between a certain number of these coefficients from which their numerical values are to be evaluated. The numerical determination of the coefficients will be facilitated when this difference-relation contains as few terms as possible. Now it is found in the present paper that, without imposing any restriction on the period of the disturbing force, if the form we assign to the terms of the series for the tide-height is that of the tesseral harmonics or LAPLACE's functions, a linear relation involving three successive coefficients only may be deduced, provided that the law of depth is such that both the internal and external surfaces of the ocean are spheroids of revolution about the polar axis. This however appears to be the most general law of depth which can be employed without obtaining more than three successive coefficients in the linear relation in question, and consequently our discussion deals only with cases where the law of depth is subject to this limitation.

In § 1 I have collected the principal properties of the functions used in the analysis. These properties are for the most part well known, but in consideration of the want of agreement in the notation employed by different writers, I have thought it best to briefly prove such of them as are required in preference to giving references to places where they may be found. Moreover I have thus been enabled to write the results in the exact form required for subsequent application.

§§ 2–4 deal with the integration of the differential equations and the deduction of the linear equations (31), (40) connecting the coefficients in the expansion of the tide-height. These equations, the analogy of which with the equations (23), (23A) of Part I. will be at once apparent, constitute the chief analytical results of the paper, and the remainder is occupied with the application of these formulæ to the discussion of the free and forced vibrations on lines similar to those adopted in Part I.

§§ 5–11 treat of the free oscillations, the discussion being confined to the case where the depth is uniform. A period-equation is obtained, and an approximate method of determining the higher roots is given. The approximations will not however be sufficiently close for the earlier roots, and consequently it is necessary to evaluate these earlier roots by trial and error. The method of procedure is indicated by numerical examples, and several of the more important roots are tabulated for four different depths of the ocean. The most interesting result is the existence of a second class of free oscillations besides those whose existence may be at once inferred by analogy from the simpler problem of the oscillations of an ocean covering a non-rotating globe. The characteristics of the oscillations of this class are discussed in § 11.

In § 12 a general analytical solution of the problem of the forced vibrations due to any disturbing force is given, but as the analytical expressions obtained are too intricate to afford much indication of the nature of the forced tides, the various types of oscillation which occur on the earth are afterwards treated numerically.

In certain cases, intimately associated with those actually occurring, the analytical expressions however admit of considerable reductions. These cases are discussed in § 14, where theorems due to LAPLACE and Professor DARWIN are obtained and generalized.

§§ 15–18 contain numerical examples of the evaluation of the semi-diurnal and diurnal tidal constituents. The arithmetic is considerably simplified when the period of the disturbing force is rigorously equal to half a sidereal day or a sidereal day, and consequently these cases are first dealt with and the results compared with those of LAPLACE. Additional examples are however also given to illustrate the effects of the departure of the periods from exact coincidence with half a sidereal day and a sidereal day respectively, the cases selected for investigation corresponding with the leading lunar constituents.

§ 1. *Properties of Tesseral Harmonics.*

Let $P_n(\mu)$ denote the zonal harmonic of order n. Then P_n is the solution which remains finite when $\mu = \pm 1$ of the differential equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_n}{d\mu}\right\} + n(n+1)P_n = 0 \quad \ldots \ldots \quad (1).$$

Let

$$P_n^s(\mu) = (1-\mu^2)^{\frac{1}{2}s}\frac{d^sP_n}{d\mu^s} \quad \ldots \ldots \quad (2).$$

Then, on differentiating the equation (1) s times, we obtain

$$(1-\mu^2)\frac{d^{s+2}P_n}{d\mu^{s+2}} - 2(s+1)\mu\frac{d^{s+1}P_n}{d\mu^{s+1}} + (n-s)(n+s+1)\frac{d^sP_n}{d\mu^s} = 0,$$

or,

$$(1-\mu^2)^{\frac{1}{2}(s+2)}\frac{d^2}{d\mu^2}\{(1-\mu^2)^{-\frac{1}{2}s}P_n^s\} - 2(s+1)\mu(1-\mu^2)^{\frac{1}{2}s}\frac{d}{d\mu}\{(1-\mu^2)^{-\frac{1}{2}s}P_n^s\}$$
$$+ (n-s)(n+s+1)P_n^s = 0,$$

which on reduction gives

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_n^s}{d\mu}\right\}+\left\{n(n+1)-\frac{s^2}{1-\mu^2}\right\}P_n^s=0 \quad . \quad . \quad . \quad . \quad (3).$$

Thus P_n^s is a solution of the equation (3); the form (2) shows that it does not become infinite when $\mu = \pm 1$, while from (3) we see that the two functions $P_n^s \cos s\phi$, $P_n^s \sin s\phi$, or what is equivalent, the two functions $P_n^s e^{\pm is\phi}$, are spherical surface-harmonics of order n. In our subsequent work the latter forms involving the imaginary exponential will be more convenient than the real trigonometrical forms. We shall therefore describe the functions $P_n^s(\mu)e^{\pm is\phi}$ as the tesseral harmonics of order n and rank s. In some cases it may be convenient to apply the same nomenclature to the "associated function" $P_n^s(\mu)$, but whenever this is done, it must be understood that an exponential factor is implied, though not expressed.

The tesseral harmonics of course include as special cases the zonal harmonics obtained by putting $s = 0$, and the sectorial harmonics obtained by putting $s = n$, while, in accordance with the definition (2), for values of s greater than n we may suppose that $P_n^s(\mu) = 0$.

The principal properties of the tesseral harmonics which we shall require may be derived from those of the zonal harmonics. Thus, if we differentiate s times the well-known relation

$$(n+1)P_{n+1}-(2n+1)\mu P_n+nP_{n-1}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

we obtain

$$(n+1)\frac{d^sP_{n+1}}{d\mu^s}-(2n+1)\mu\frac{d^sP_n}{d\mu^s}-(2n+1)s\frac{d^{s-1}P_n}{d\mu^{s-1}}+n\frac{d^sP_{n-1}}{d\mu^s}=0,$$

which, on making use of the formula

$$\frac{dP_{n+1}}{d\mu}-\frac{dP_{n-1}}{d\mu}=(2n+1)P_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

gives

$$(n-s+1)\frac{d^sP_{n+1}}{d\mu^s}-(2n+1)\mu\frac{d^sP_n}{d\mu^s}+(n+s)\frac{d^sP_{n-1}}{d\mu^s}=0.$$

On multiplying by the factor $(1-\mu^2)^{\frac{1}{2}s}$ this may be written

$$(n-s+1)P_{n+1}^s-(2n+1)\mu P_n^s+(n+s)P_{n-1}^s=0 \quad . \quad . \quad . \quad (6).$$

Again by differentiating the equation (2) we find

$$\begin{aligned}(1-\mu^2)\frac{dP_n^s}{d\mu} &= -s\mu(1-\mu^2)^{\frac{1}{2}s}\frac{d^sP_n}{d\mu^s}+(1-\mu^2)^{\frac{1}{2}s+1}\frac{d^{s+1}P_n}{d\mu^{s+1}}\\ &= -s\mu P_n^s+(1-\mu^2)^{\frac{1}{2}s}\left[\frac{d^s}{d\mu^s}\left\{(1-\mu^2)\frac{dP_n}{d\mu}\right\}+2s\mu\frac{d^sP_n}{d\mu^s}+s(s-1)\frac{d^{s-1}P_n}{d\mu^{s-1}}\right]\\ &= s\mu P_n^s-(1-\mu^2)^{\frac{1}{2}s}\left[n(n+1)\frac{d^{s-1}P_n}{d\mu^{s-1}}-s(s-1)\frac{d^{s-1}P_n}{d\mu^{s-1}}\right]\end{aligned}$$

in virtue of the differential equation (1) for P_n; and therefore by means of (5)

$$(1-\mu^2)\frac{dP_n^s}{d\mu} = s\mu P_n^s - \frac{(n-s+1)(n+s)}{2n+1}(1-\mu^2)^{\frac{1}{2}s}\frac{d^s}{d\mu^s}\{P_{n+1} - P_{n-1}\}$$

$$= s\mu P_n^s - \frac{(n-s+1)(n+s)}{2n+1}(P_{n+1}^s - P_{n-1}^s),$$

which with the aid of (6) may be expressed in the form

$$(1-\mu^2)\frac{dP_n^s}{d\mu} = -\frac{n(n-s+1)}{2n+1}P_{n+1}^s + \frac{(n+1)(n+s)}{2n+1}P_{n-1}^s \quad . \quad . \quad . \quad (7).$$

Let us write for brevity

$$\left.\begin{aligned} D &\equiv (1-\mu^2)\frac{d}{d\mu} \\ \Delta &\equiv \frac{d}{d\mu}(D) - \frac{s^2}{1-\mu^2} = \frac{1}{1-\mu^2}(D^2 - s^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Then the equation (3) may be written

$$\Delta P_n^s = -n(n+1)P_n^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

while, if σ denote any constant quantity, we obtain from (6), (7)

$$(D + \sigma\mu)P_n^s = -\frac{(n-\sigma)(n-s+1)}{2n+1}P_{n+1}^s + \frac{(n+\sigma+1)(n+s)}{2n+1}P_{n-1}^s \quad . \quad (10).$$

The relation between the operators D, Δ may be written in the forms

$$\left.\begin{aligned} (D-\sigma\mu)(D+\sigma\mu) - (s^2 - \sigma^2\mu^2) &= (1-\mu^2)(\Delta + \sigma) \\ (D+\sigma\mu)(D-\sigma\mu) - (s^2 - \sigma^2\mu^2) &= (1-\mu^2)(\Delta - \sigma) \end{aligned}\right\} \quad . \quad . \quad (11),$$

which will be useful hereafter.

§ 2. *Transformation of the Dynamical Equations for the Tides.*

The formation of the differential equations for the tidal oscillations of the ocean has been fully dealt with in Part I. It is there shown (§ 4) that, if U, V denote the northward and eastward velocity-components in latitude $\sin^{-1}\mu$ and longitude ϕ, when the system is executing a simple harmonic vibration in period $2\pi/\lambda$, these velocity-components will be expressible in terms of a single function ψ by means of the equations

$$\left.\begin{aligned} U &= -\frac{i\lambda\sqrt{(1-\mu^2)}}{a(\lambda^2 - 4\omega^2\mu^2)}\frac{\partial\psi}{\partial\mu} + \frac{2\omega\mu}{a\sqrt{(1-\mu^2)}(\lambda^2 - 4\omega^2\mu^2)}\frac{\partial\psi}{\partial\phi} \\ V &= -\frac{2\omega\mu\sqrt{(1-\mu^2)}}{a(\lambda^2 - 4\omega^2\mu^2)}\frac{\partial\psi}{\partial\mu} - \frac{i\lambda}{a\sqrt{(1-\mu^2)}(\lambda^2 - 4\omega^2\mu^2)}\frac{\partial\psi}{\partial\phi} \end{aligned}\right\} \quad . \quad . \quad (12).$$

Supposing that U, V, ψ are each proportional to $e^{i(\lambda t+s\phi)}$, we may take $\partial\psi/\partial\phi = is\psi$, and therefore, if we put $2\omega s/\lambda = \sigma$ and introduce the abridged notation of the previous section, we may write the above equations in the form—

$$\left.\begin{aligned}\sqrt{(1-\mu^2)}\,\mathrm{U} &= -\frac{is\sigma}{2\omega a}\frac{1}{s^2-\sigma^2\mu^2}(\mathrm{D}-\sigma\mu)\psi\\ \sqrt{(1-\mu^2)}\,\mathrm{V} &= -\frac{\sigma^2}{2\omega a}\left[\frac{1}{s^2-\sigma^2\mu^2}\mu\mathrm{D}-\frac{s^2}{\sigma(s^2-\sigma^2\mu^2)}\right]\psi\\ &= -\frac{\sigma^2}{2\omega a}\left[\frac{1}{s^2-\sigma^2\mu^2}\mu(\mathrm{D}-\sigma\mu)-\frac{1}{\sigma}\right]\psi\end{aligned}\right\} \quad . \quad . \quad . \quad (13).$$

The equation of continuity is

$$\frac{\partial\zeta}{\partial t} = -\frac{1}{a}\left[\frac{\partial}{\partial\mu}\{\sqrt{(1-\mu^2)}\,h\mathrm{U}\}+\frac{\partial}{\partial\phi}\left\{\frac{h\mathrm{V}}{\sqrt{(1-\mu^2)}}\right\}\right]$$

where h denotes the depth, and ζ the height of the surface-waves. On substituting for U, V from (13) and performing the differentiations with regard to t, ϕ this becomes

$$(1-\mu^2)\zeta = \frac{\sigma^2}{4\omega^2a^2}\left[\mathrm{D}\left\{\frac{h}{s^2-\sigma^2\mu^2}(\mathrm{D}-\sigma\mu)\psi\right\}\right]+\frac{\sigma^3}{4\omega^2a^2}\left[\frac{h\mu}{s^2-\sigma^2\mu^2}(\mathrm{D}-\sigma\mu)\psi\right]-\frac{h\sigma^2}{4\omega^2a^2}\psi,$$

or

$$\frac{4\omega^2a^2}{\sigma^2}(1-\mu^2)\zeta = (\mathrm{D}+\sigma\mu)\left[\frac{h}{s^2-\sigma^2\mu^2}(\mathrm{D}-\sigma\mu)\psi\right]-h\psi \quad . \quad . \quad . \quad (14).$$

This equation is equivalent to the equation (17) of Part I. A second equation for the determination of the two functions ψ, ζ is obtained from the pressure-condition at the free surface. On reference to § 2 of Part I., this condition is seen to lead to

$$\psi = v' - g\zeta + v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

where v' denotes the surface-value of the potential due to the harmonic inequalities, and v the surface-value of the disturbing potential.

In order to effect the integration of these equations, we introduce two auxiliary functions Ψ_1, Ψ_2, connected with ψ by the relation

$$\psi = (\mathrm{D}+\sigma\mu)\Psi_1 + (s^2-\sigma^2\mu^2)\Psi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

On applying the operator $(\mathrm{D}-\sigma\mu)$ to the two members of this equation, we obtain in virtue of (11)

$$\begin{aligned}(\mathrm{D}-\sigma\mu)\psi = (1-\mu^2)(\Delta+\sigma)\Psi_1 &+ (s^2-\sigma^2\mu^2)\Psi_1\\ &+ (s^2-\sigma^2\mu^2)(\mathrm{D}-\sigma\mu)\Psi_2 - 2\sigma^2\mu(1-\mu^2)\Psi_2 \quad . \quad . \quad (17).\end{aligned}$$

Now the functions Ψ_1, Ψ_2 have as yet been subjected only to the single condition

(16). We may therefore impose on them any other arbitrary condition not inconsistent with the former. Suppose we choose them so as to satisfy the relation

$$(\Delta + \sigma)\Psi_1 = 2\sigma^2\mu\Psi_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

The two equations (16), (18) serve for the complete definition of the two functions Ψ_1, Ψ_2; making use of the latter, (17) reduces to

$$(\mathrm{D} - \sigma\mu)\psi = (s^2 - \sigma^2\mu^2)\{\Psi_1 + (\mathrm{D} - \sigma\mu)\Psi_2\} \quad . \quad . \quad . \quad . \quad (19).$$

Thus on replacing ψ by its value in terms of Ψ_1, Ψ_2 in the right-hand member of (14), we deduce

$$\frac{4\omega^2 a^2}{\sigma^2}(1-\mu^2)\zeta = (\mathrm{D}+\sigma\mu)\{h\Psi_1 + h(\mathrm{D}-\sigma\mu)\Psi_2\} - h[(\mathrm{D}+\sigma\mu)\Psi_1 + (s^2-\sigma^2\mu^2)\Psi_2].$$

If we suppose that h is constant, the terms involving Ψ_1 will disappear, while in virtue of (11) we shall obtain

$$\frac{4\omega^2 a^2}{\sigma^2}(1-\mu^2)\zeta = h(1-\mu^2)(\Delta - \sigma)\Psi_2$$

or

$$(\Delta - \sigma)\Psi_2 = \frac{4\omega^2 a^2}{\sigma^2 h}\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

We have now for the determination of the functions ψ, ζ, Ψ_1, Ψ_2 the four simultaneous differential equations (15), (16), (18), (20).

§ 3. *Integration in Series of Tesseral Harmonics.*

Let us suppose that ψ, ζ, v, Ψ_1, Ψ_2 are each expressible as series of tesseral harmonics of the same rank s. Omitting the exponential factor $e^{i(\lambda t + s\phi)}$, we assume that

$$\left.\begin{aligned}
\psi &= \sum_{n=s}^{n=\infty} \Gamma_n^s \mathrm{P}_n^s(\mu),\\
\zeta &= \sum_{n=s}^{n=\infty} \mathrm{C}_n^s \mathrm{P}_n^s(\mu),\\
v &= \sum_{n=s}^{n=\infty} \gamma_n^s \mathrm{P}_n^s(\mu),\\
\Psi_1 &= \sum_{n=s}^{n=\infty} \alpha_n^s \mathrm{P}_n^s(\mu),\\
\Psi_2 &= \sum_{n=s}^{n=\infty} \beta_n^s \mathrm{P}_n^s(\mu),
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

Then, if ρ denote the density of the water, and σ_0 the mean density of the whole system inclusive of the ocean, by well-known properties of surface-harmonics it may be shown that

$$v' = \Sigma C_n^s \frac{3\rho g}{(2n+1)\sigma_0} P_n^s.$$

Thus from (15), on replacing the quantities involved by means of their expansions in terms of associated functions, we obtain

$$\Sigma \Gamma_n^s P_n^s(\mu) = -g\Sigma C_n^s \left\{1 - \frac{3\rho}{(2n+1)\sigma_0}\right\} P_n^s(\mu) + \Sigma\gamma_n^s P_n^s(\mu);$$

whence, if we equate coefficients of P_n^s in the two members, we deduce

$$\Gamma_n^s = -g_n C_n^s + \gamma_n^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where we have written for brevity

$$g_n \equiv g\left\{1 - \frac{3\rho}{(2n+1)\sigma_0}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

From (16), (18), we have

$$\psi = (D + \sigma\mu)\Psi_1 + s^2\Psi_2 - \tfrac{1}{2}\mu(\Delta + \sigma)\Psi_1 \quad . \quad . \quad . \quad . \quad . \quad (24).$$

But by means of (10) we find, on replacing Ψ_1 by its expansion,

$$(D + \sigma\mu)\Psi_1 = -\Sigma\alpha_n^s\left[\frac{(n-\sigma)(n-s+1)}{2n+1}P_{n+1}^s - \frac{(n+\sigma+1)(n+s)}{2n+1}P_{n-1}^s\right],$$

while from (9), (6), we obtain

$$\mu(\Delta + \sigma)\Psi_1 = \Sigma\alpha_n^s\{\sigma - n(n+1)\}\left[\frac{n-s+1}{2n+1}P_{n+1}^s + \frac{n+s}{2n+1}P_{n-1}^s\right].$$

Thus

$$[(D + \sigma\mu) - \tfrac{1}{2}\mu(\Delta + \sigma)]\Psi_1$$
$$= \tfrac{1}{2}\Sigma\alpha_n^s\left[\frac{\{n(n-1)+\sigma\}(n-s+1)}{2n+1}P_{n+1}^s + \frac{\{(n+1)(n+2)+\sigma\}(n+s)}{2n+1}P_{n-1}^s\right],$$

and the right-hand member of (24) is therefore equal to

$$\Sigma s^2\beta_n^s P_n^s$$
$$+ \tfrac{1}{2}\Sigma\alpha_n^s\left[\frac{\{n(n-1)+\sigma\}(n-s+1)}{2n+1}P_{n+1}^s + \frac{\{(n+1)(n+2)+\sigma\}(n+s)}{2n+1}P_{n-1}^s\right].$$

Hence, on comparing the coefficients of P_n^s in the two members of (24), we obtain

$$\Gamma_n^s = s^2\beta_n^s + \tfrac{1}{2}\frac{(n-s)\{(n-1)(n-2)+\sigma\}}{2n-1}\alpha_{n-1}^s + \tfrac{1}{2}\frac{(n+s+1)\{(n+2)(n+3)+\sigma\}}{2n+3}\alpha_{n+1}^s \quad (25).$$

Again, on replacing Ψ_1, Ψ_2 by their expansions in (18), the two members may be expressed as series of associated functions by means of (9), (6). Thus we find

$$-\Sigma\alpha_n^s\{n(n+1)-\sigma\}P_n^s = 2\sigma^2\Sigma\beta_n^s\left[\frac{n-s+1}{2n+1}P_{n+1}^s + \frac{n+s}{2n+1}P_{n-1}^s\right],$$

whence, on equating coefficients, we deduce that

$$\alpha_n^s\{n(n+1)-\sigma\} = -2\sigma^2\left\{\frac{n-s}{2n-1}\beta_{n-1}^s + \frac{n+s+1}{2n+3}\beta_{n+1}^s\right\} \quad . \quad . \quad (26).$$

Finally from (20), on expressing the two members by their expansions and equating coefficients, we obtain

$$\{n(n+1)+\sigma\}\beta_n^s = -\frac{4\omega^2a^2}{\sigma^2h}C_n^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

The relations (26), (27), enable us to eliminate the auxiliary constants β_n^s, α_{n-1}^s, α_{n+1}^s from (25), and thus to express Γ_n^s as a linear function of C_{n-2}^s, C_n^s, C_{n+2}^s. On substituting for α_{n-1}^s, α_{n+1}^s from the formula (26) in (25), we obtain

$$\Gamma_n^s = s^2\beta_n^s - \sigma^2\frac{(n-s)\{(n-1)(n-2)+\sigma\}}{(2n-1)\{(n-1)n-\sigma\}}\left\{\frac{n-s-1}{2n-3}\beta_{n-2}^s + \frac{n+s}{2n+1}\beta_n^s\right\}$$
$$-\sigma^2\frac{(n+s+1)\{(n+2)(n+3)+\sigma\}}{(2n+3)\{(n+1)(n+2)-\sigma\}}\left\{\frac{n-s+1}{2n+1}\beta_n^s + \frac{n+s+2}{2n+5}\beta_{n+2}^s\right\}$$
$$= -\sigma^2\frac{(n-s)(n-s-1)\{(n-1)(n-2)+\sigma\}}{(2n-1)(2n-3)\{(n-1)n-\sigma\}}\beta_{n-2}^s$$
$$+\left[s^2-\sigma^2\frac{(n-s)(n+s)\{(n-1)(n-2)+\sigma\}}{(2n-1)(2n+1)\{(n-1)n-\sigma\}}-\sigma^2\frac{(n-s+1)(n+s+1)\{(n+2)(n+3)+\sigma\}}{(2n+1)(2n+3)\{(n+1)(n+2)-\sigma\}}\right]\beta_n^s$$
$$-\sigma^2\frac{(n+s+1)(n+s+2)\{(n+2)(n+3)+\sigma\}}{(2n+3)(2n+5)\{(n+1)(n+2)-\sigma\}}\beta_{n+2}^s,$$

and this, by means of (27), gives

$$\frac{h\Gamma_n^s}{4\omega^2a^2} = \frac{(n-s)(n-s-1)}{(2n-1)(2n-3)\{(n-1)n-\sigma\}}C_{n-2}^s - \Lambda_n^s C_n^s$$
$$+\frac{(n+s+1)(n+s+2)}{(2n+3)(2n+5)\{(n+1)(n+2)-\sigma\}}C_{n+2}^s \quad . \quad . \quad (28),$$

where

$$\Lambda_n^s = \frac{s^2}{\sigma^2\{n(n+1)+\sigma\}} - \frac{(n-s)(n+s)\{(n-1)(n-2)+\sigma\}}{(2n-1)(2n+1)\{(n-1)n-\sigma\}\{n(n+1)+\sigma\}}$$
$$-\frac{(n+s+1)(n-s+1)\{(n+2)(n+3)+\sigma\}}{(2n+1)(2n+3)\{(n+1)(n+2)-\sigma\}\{n(n+1)+\sigma\}}.$$

This expression for Λ_n^s may be somewhat simplified if we separate it into its component partial fractions; we thus find

$$\Lambda_n^s = \frac{s^2}{\sigma^2\{n(n+1)+\sigma\}} - \frac{(n-s)(n+s)}{(2n-1)(2n+1)}\left[\frac{(n-1)^2}{n^2\{n(n-1)-\sigma\}} - \frac{2n-1}{n^2\{n(n+1)+\sigma\}}\right]$$
$$- \frac{(n-s+1)(n+s+1)}{(2n+1)(2n+3)}\left[\frac{(n+2)^2}{(n+1)^2\{(n+1)(n+2)-\sigma\}} + \frac{2n+3}{(n+1)^2\{n(n+1)+\sigma\}}\right]$$
$$= \frac{1}{\{n(n+1)+\sigma\}}\left[\frac{s^2}{\sigma^2} + \frac{n^2-s^2}{n^2(2n+1)} - \frac{(n+1)^2-s^2}{(n+1)^2(2n+1)}\right]$$
$$- \frac{(n-1)^2(n-s)(n+s)}{n^2(2n-1)(2n+1)\{n(n-1)-\sigma\}} - \frac{(n+2)^2(n-s+1)(n+s+1)}{(n+1)^2(2n+1)(2n+3)\{(n+1)(n+2)-\sigma\}};$$

whence finally, remembering that $\sigma = 2\omega s/\lambda$,

$$\Lambda_n^s = \frac{\lambda^2}{4\omega^2}\frac{n(n+1)-2\omega s/\lambda}{n^2(n+1)^2} - \frac{(n-1)^2(n-s)(n+s)}{n^2(2n-1)(2n+1)\{(n-1)n-2\omega s/\lambda\}}$$
$$- \frac{(n+2)^2(n-s+1)(n+s+1)}{(n+1)^2(2n+1)(2n+3)\{(n+1)(n+2)-2\omega s/\lambda\}} \quad . \quad (29).$$

The relation (28) will hold for all values of n equal to or greater than s, provided we suppose that $C_{s-2}^s = 0$ and $C_{s-1}^s = 0$. If we put for brevity

$$\left.\begin{aligned} x_n^s &= \frac{(n-s+1)(n-s+2)}{(2n+1)(2n+3)\{(n+1)(n+2)-2\omega s/\lambda\}}, \\ y_n^s &= \frac{(n+s+1)(n+s+2)}{(2n+3)(2n+5)\{(n+1)(n+2)-2\omega s/\lambda\}} \end{aligned}\right\} \quad . \quad . \quad . \quad (30),$$

it may be written

$$\frac{h\Gamma_n^s}{4\omega^2 a^2} = x_{n-2}^s C_{n-2}^s - \Lambda_n^s C_n^s + y_n^s C_{n+2}^s.$$

Replacing Γ_n^s by its value in terms of C_n^s, γ_n^s [equation (22)], we obtain

$$x_{n-2}^s C_{n-2}^s - L_n^s C_n^s + y_n^s C_{n+2}^s = \frac{h\gamma_n^s}{4\omega^2 a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

where x_{n-2}^s, y_n^s are defined by (30) and

$$L_n^s = -\frac{hg_n}{4\omega^2 a^2} + \Lambda_n^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

Λ_n^s being defined by the equation (29).

On putting $s = 0$ it may readily be verified that the equation (31) reduces to the equation (23) of Part I. The manner in which such an equation may be utilized for the determination of the free and forced vibrations has been fully discussed in that paper. A strictly analogous procedure might be adopted in the present case, but it is found convenient to modify the previous treatment in some respects. We shall therefore only indicate briefly the course of procedure when our analysis corresponds with that given in Part I., giving greater detail as regards those points where a different method has been found desirable.

§ 4. *Extension to the Case of Variable Depth.*

The formulæ developed in the preceding sections apply only to the case where the depth is uniform. We may however obtain a relation of the same nature as the equation (31) when the depth is a function of the latitude given by the formula

$$h = k + l(1 - \mu^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

that is, when both the internal and external surfaces of the ocean are spheroids of revolution.

The above expression for h may be written in the form

$$h = \kappa + l\left(\frac{\lambda^2}{4\omega^2} - \mu^2\right) = \kappa + \frac{l(s^2 - \sigma^2\mu^2)}{\sigma^2} \quad . \quad . \quad . \quad . \quad . \quad (34),$$

where

$$\kappa = k + l\left(1 - \frac{\lambda^2}{4\omega^2}\right) . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

Substituting the expression (34) for h in the equation (14) we obtain

$$\frac{4\omega^2 a^2}{\sigma^2}(1 - \mu^2)\,\zeta = (\mathrm{D} + \sigma\mu)\left[\frac{\kappa}{s^2 - \sigma^2\mu^2}(\mathrm{D} - \sigma\mu)\,\psi\right] - \kappa\psi$$
$$+ \frac{l}{\sigma^2}\left[(\mathrm{D} + \sigma\mu)(\mathrm{D} - \sigma\mu) - (s^2 - \sigma^2\mu^2)\right]\psi,$$

which, with the aid of (11), becomes

$$\frac{4\omega^2 a^2}{\sigma^2}(1 - \mu^2)\left[\zeta - \frac{l}{4\omega^2 a^2}(\Delta - \sigma)\,\psi\right] = (\mathrm{D} + \sigma\mu)\left[\frac{\kappa}{\sigma^2 - \sigma^2\mu^2}(\mathrm{D} - \sigma\mu)\,\psi\right] - \kappa\psi.$$

The right-hand member is of the same form as that of the equation (14), except that h is replaced by κ. We may therefore introduce two auxiliary functions Ψ_1, Ψ_2, defined by (16), (18), and proceed as in § 2, and we shall obtain in place of (20) the equation

$$(\Delta - \sigma)\,\Psi_2 = \frac{4\omega^2 a^2}{\sigma^2 \kappa}\left[\zeta - \frac{l}{4\omega^2 a^2}(\Delta - \sigma)\,\psi\right] \quad . \quad . \quad . \quad . \quad . \quad (36).$$

But, if we introduce the expansions (21) for ψ, ζ, we find

$$\zeta - \frac{l}{4\omega^2 a^2}(\Delta - \sigma)\psi = \Sigma\left[C_n^s + \frac{l}{4\omega^2 a^2}\{n(n+1)+\sigma\}\Gamma_n^s\right]P_n^s \quad . \quad . \quad (37),$$

whence the final equation which replaces (31) will be obtained by replacing C_n^s by

$$C_n^s + \frac{l}{4\omega^2 a^2}\{n(n+1)+\sigma\}\Gamma_n^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

in the right-hand member of (28), and h by κ in the left-hand member.

Thus we obtain

$$\begin{aligned}\frac{\kappa\Gamma_n^s}{4\omega^2 a^2} = x_{n-2}^s&\left[C_{n-2}^s + \frac{l}{4\omega^2 a^2}\{(n-2)(n-1)+\sigma\}\Gamma_{n-2}^s\right]\\ &- \Lambda_n^s\left[C_n^s + \frac{l}{4\omega^2 a^2}\{n(n+1)+\sigma\}\Gamma_n^s\right]\\ &+ y_n^s\left[C_{n+2}^s + \frac{l}{4\omega^2 a^2}\{(n+2)(n+3)+\sigma\}\Gamma_{n+2}^s\right] \quad . \quad . \quad (39);\end{aligned}$$

and therefore, on separating out the parts of Γ_n^s due to C_n^s, γ_n^s respectively, we find

$$\xi_{n-2}^s C_{n-2}^s - \mathfrak{L}_n^s C_n^s + \eta_n^s C_{n+2}^s = G_n^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40),$$

where

$$\left.\begin{aligned}\xi_n^s &= x_n^s\left[1 - \frac{lg_n}{4\omega^2 a^2}\left\{n(n+1)+\frac{2\omega s}{\lambda}\right\}\right]\\ \eta_n^s &= y_n^s\left[1 - \frac{lg_{n+2}}{4\omega^2 a^2}\left\{(n+2)(n+3)+\frac{2\omega s}{\lambda}\right\}\right]\\ \mathfrak{L}_n^s &= -\frac{\kappa g_n}{4\omega^2 a^2} + \Lambda_n^s\left[1 - \frac{lg_n}{4\omega^2 a^2}\left\{n(n+1)+\frac{2\omega s}{\lambda}\right\}\right]\end{aligned}\right\} \quad . \quad . \quad . \quad (41),$$

and

$$\begin{aligned}G_n^s = &-\frac{l}{4\omega^2 a^2}\left[(n-2)(n-1)+\frac{2\omega s}{\lambda}\right]x_{n-2}^s\gamma_{n-2}^s\\ &+\left[\frac{\kappa}{4\omega^2 a^2} + \frac{l}{4\omega^2 a^2}\left\{n(n+1)+\frac{2\omega s}{\lambda}\right\}\Lambda_n^s\right]\gamma_n^s\\ &-\frac{l}{4\omega^2 a^2}\left\{(n+2)(n+3)+\frac{2\omega s}{\lambda}\right\}y_n^s\gamma_{n+2}^s \quad . \quad . \quad . \quad . \quad . \quad (42).\end{aligned}$$

For the special case $s = 0$ the equation (40) reduces to the equation (23A) of Part I.

§ 5. *The Period-equation for the Free Oscillations in an Ocean of Uniform Depth.*

To determine the periods of free oscillation we may proceed exactly as in § 6 of

Part I., making use of the equation (31) instead of the simpler equation (23) of the previous paper. On putting γ_n^s zero, the equation (31) gives

$$x_{n-2}^s C_{n-2}^s - L_n^s C_n^s + y_n^s C_{n+2}^s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43),$$

an equation which must hold for all values of n equal to or greater than s, it being understood that $C_{s-2}^s = 0$, and $C_{s-1}^s = 0$. The series of equations typified by (43) may be divided into two groups, in the former of which the suffixes involved are such that $n - s$ is even and in the latter odd. The types of motion resulting from these two groups may be treated independently, the former being characterised by symmetry with respect to the equator and the latter by asymmetry. The treatment of the two groups of equations will be exactly similar, and we shall therefore in the main confine our discussion to the former group.

If we introduce the notation

$$\left.\begin{aligned} H_n^s &= \frac{x_n^s y_n^s}{L_n^s} - \frac{x_{n-2}^s y_{n-2}^s}{L_{n-2}^s} - \ldots - \frac{x_s^s y_s^s}{L_s^s} \\ K_n^s &= \frac{x_{n-2}^s y_{n-2}^s}{L_n^s} - \frac{x_n^s y_n^s}{L_{n+2}^s} - \ldots \textit{ad inf.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44),$$

it may be shown as in Part I. that provided $\underset{n=\infty}{L^t} C_n^s = 0$,

$$\frac{x_n^s C_n^s}{C_{n+2}^s} = H_n^s, \qquad \frac{y_{n-2}^s C_n^s}{C_{n-2}^s} = K_n^s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45),$$

and therefore the equation (43) may be written

$$C_n^s [H_{n-2}^s - L_n^s + K_{n+2}^s] = 0,$$

whence the period-equation for the free oscillations of symmetrical type is obtainable in the form

$$L_n^s - H_{n-2}^s - K_{n+2}^s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46),$$

when $n - s$ is an even integer.

The same equation will apply to the asymmetrical types if we suppose that $n - s$ is an odd integer, and that the continued fraction H_n^s terminates with the partial quotient $\dfrac{x_{s+1}^s y_{s+1}^s}{L_{s+1}^s}$.

In particular, putting $n - s = 0$, we can express the period-equation for the symmetrical types in the form

$$L_s^s - \frac{x_s^s y_s^s}{L_{s+2}^s} - \frac{x_{s+2}^s y_{s+2}^s}{L_{s+4}^s} - \ldots \textit{ad inf.} = 0 \quad . \quad . \quad . \quad . \quad . \quad (46a),$$

while, putting $n - s = 1$, that for the asymmetrical types may be written

$$L_{s+1}^s - \frac{x_{s+1}^s y_{s+1}^s}{L_{s+3}^s} - \frac{x_{s+3}^s y_{s+3}^s}{L_{s+5}^s} - \ldots \textit{ad inf.} = 0 \quad . \quad . \quad . \quad . \quad . \quad (46b).$$

On the analogy of the problem dealt with in Part I., we may anticipate that, when λ has a value in the neighbourhood of a root of the equation $L_n^s = 0$, the continued fractions H_{n-2}^s, K_{n+2}^s will rapidly converge to small values. Further, with large values of n, the numerical values of H_{n-2}^s, K_{n+2}^s tend to become equal with opposite signs. Hence there will be roots of the equation (46) which approximate to roots of the equation

$$L_n^s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47).$$

Let us therefore examine the nature of the roots of this latter equation; this may best be done by considering the graph of the function Λ_n^s. Putting $y = \Lambda_n^s$, $x = \lambda/\omega$, we have to consider the form of the curve

$$y = \tfrac{1}{4} x^2 \frac{n(n+1) - 2s/x}{n^2(n+1)^2} - \frac{(n-1)^2(n-s)(n+s)}{n^2(2n-1)(2n+1)\{n(n-1) - 2s/x\}} - \frac{(n+2)^2(n-s+1)(n+s+1)}{(n+1)^2(2n+1)(2n+3)\{(n+1)(n+2) - 2s/x\}}.$$

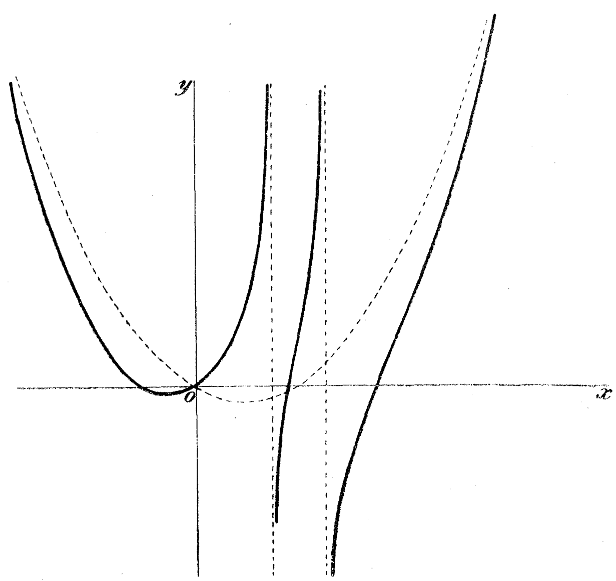
It is evident that this curve will have two rectilinear asymptotes parallel to the axis of y, whose equations are

$$x = \frac{2s}{n(n-1)}, \qquad x = \frac{2s}{(n+1)(n+2)},$$

and that on passing these critical values, with increasing x, the sign of y will change from positive to negative. The curve will pass through the origin, while when x is very large it will approximate to the parabola

$$y = \tfrac{1}{4} x^2 \frac{n(n+1) - 2s/x}{n^2(n+1)^2}.$$

Hence it must consist of three branches as in the annexed diagram, where the dotted lines represent the rectilinear and parabolic asymptotes.



The roots of the equation $L_n^s = 0$, regarded as an equation for the determination of λ/ω, will be the abscissæ of the points of intersection of this curve with the line

$$y = hg_n/4\omega^2 a^2.$$

Since h is essentially positive, the roots will all be real, and they will lie in the intervals between

$$-\infty, \quad 0, \quad \frac{2s}{(n+1)(n+2)}, \quad \frac{2s}{n(n-1)}, \quad +\infty.$$

For large values of $hg_n/4\omega^2 a^2$ the two extreme roots will approximate to the roots of the equation

$$-\frac{hg_n}{4\omega^2 a^2} + \frac{\lambda^2}{4\omega^2}\frac{n(n+1) - 2\omega s/\lambda}{n^2(n+1)^2} = 0,$$

while the remaining two roots will approximate to

$$\frac{2s}{(n+1)(n+2)}, \quad \frac{2s}{n(n-1)}.$$

The two former roots are those which have their analogue in the special case treated of in Part I., for which $s = 0$, in which case they have equal magnitudes but opposite signs. These roots, we may expect, will approximate to roots of the period-equation, at least when n is large.

In order to see the significance of the remaining roots, it is convenient to transform the period-equation into a different form, which moreover is far better adapted for the more accurate numerical determination of the earlier roots.

§ 6. *Modified Form of the Period-Equation.*

Referring back to the equations (29), (30), (32), which define x_n^s, y_n^s, L_n^s, we see that (31) may be written in the form

$$\begin{aligned}&\frac{(n-s)}{(2n-1)\{n(n-1)-2\omega s/\lambda\}}\left[\frac{(n-s-1)}{(2n-3)}C_{n-2}^s + \frac{(n-1)^2(n+s)}{n^2(2n+1)}C_n^s\right]\\ &\quad -\left[\frac{\lambda^2}{4\omega^2}\frac{n(n+1)-2\omega s/\lambda}{n^2(n+1)^2} - \frac{hg_n}{4\omega^2a^2}\right]C_n^s\\ &\quad + \frac{(n+s+1)}{(2n+3)\{(n+1)(n+2)-2\omega s/\lambda\}}\left[\frac{(n+2)^2(n-s+1)}{(n+1)^2(2n+1)}C_n^s + \frac{(n+s+2)}{(2n+5)}C_{n+2}^s\right] = \frac{h\gamma_n^s}{4\omega^2a^2} \quad (48).\end{aligned}$$

Hence, if we introduce a new set of auxiliary constants, D_s^s, D_{s+1}^s, &c., such that for values of n equal to or greater than s

$$\frac{(n+1)^2(n-s)}{2n-1}C_{n-1}^s + \frac{n^2(n+s+1)}{2n+3}C_{n+1}^s = \left\{n(n+1) - \frac{2\omega s}{\lambda}\right\}D_n^s \quad . \quad (49),$$

the equation (48) reduces to

$$\frac{(n+1)^2(n-s)}{2n-1}D^s_{n-1} + \frac{n^2(n+s+1)}{2n+3}D^s_{n+1}$$
$$= \left[\frac{\lambda^2}{4\omega^2}\left\{n(n+1) - \frac{2\omega s}{\lambda}\right\} - n^2(n+1)^2\frac{hg_n}{4\omega^2a^2}\right]C^s_n + n^2(n+1)^2\frac{h\gamma^s_n}{4\omega^2a^2} \quad (50).$$

Thus, if we write for brevity

$$\left.\begin{aligned} M^s_n &= \frac{\lambda^2}{4\omega^2}\left\{n(n+1) - \frac{2\omega s}{\lambda}\right\} - n^2(n+1)^2\frac{hg_n}{4\omega^2a^2} \\ N^s_n &= n(n+1) - \frac{2\omega s}{\lambda} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (51),$$

and put $\gamma^s_n = 0$, the equation (43) is replaced by the two following:

$$\left.\begin{aligned} \frac{(n+1)^2(n-s)}{2n-1}D^s_{n-1} - M^s_n C^s_n + \frac{n^2(n+s+1)}{2n+3}D^s_{n+1} = 0 \\ \frac{(n+1)^2(n-s)}{2n-1}C^s_{n-1} - N^s_n D^s_n + \frac{n^2(n+s+1)}{2n+3}C^s_{n+1} = 0 \end{aligned}\right\} \quad . \quad . \quad (52).$$

For the determination of the symmetrical types we therefore have the series of equations

$$-M^s_s C^s_s + \frac{s^2(2s+1)}{2s+3}D^s_{s+1} = 0,$$

$$\frac{(s+2)^2.1}{2s+1}C^s_s - N^s_{s+1}D^s_{s+1} + \frac{(s+1)^2(2s+2)}{2s+5}C^s_{s+2} = 0,$$

$$\frac{(s+3)^2.2}{2s+3}D^s_{s+1} - M^s_{s+2}C^s_{s+2} + \frac{(s+2)^2(2s+3)}{2s+7}D^s_{s+3} = 0.$$

$$. \quad . \quad . \quad . \quad .$$

On eliminating the quantities C^s_s, D^s_{s+1}, C^s_{s+2}, &c., by means of a continued fraction we find the period-equation in the form

$$M^s_s - \frac{a^s_s}{N^s_{s+1} -}\ \frac{a^s_{s+1}}{M^s_{s+2} -}\ \frac{a^s_{s+2}}{N^s_{s+3} -} \ldots \textit{ad inf.} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (53),$$

where we have written for brevity a^s_n in place of

$$\frac{n^2(n+2)^2(n-s+1)(n+s+1)}{(2n+1)(2n+3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54).$$

In like manner the period-equation for the asymmetrical types may be written

$$N^s_s - \frac{a^s_s}{M^s_{s+1} -}\ \frac{a^s_{s+1}}{N^s_{s+2} -}\ \frac{a^s_{s+2}}{M^s_{s+3} -} \ldots \textit{ad inf.} = 0 \quad . \quad . \quad . \quad . \quad . \quad (55).$$

We may also write these equations in a variety of alternative forms in which prominence is given to any one we please of the quantities M^s_n, N^s_n. These forms are obtained by giving n different integral values in the equations

$$M^s_n - \left[\frac{a^s_{n-1}}{N^s_{n-1}} - \frac{a^s_{n-2}}{M^s_{n-2}} - \frac{a^s_{n-3}}{N^s_{n-3}} - \ldots\right] - \left[\frac{a^s_n}{N^s_{n+1}} - \frac{a^s_{n+1}}{M^s_{n+2}} - \frac{a^s_{n+2}}{N^s_{n+3}} - \ldots\right] = 0\,. \quad (56),$$

$$N^s_n - \left[\frac{a^s_{n-1}}{M^s_{n-1}} - \frac{a^s_{n-2}}{N^s_{n-2}} - \frac{a^s_{n-3}}{M^s_{n-3}} - \ldots\right] - \left[\frac{a^s_n}{M^s_{n+1}} - \frac{a^s_{n+1}}{N^s_{n+2}} - \frac{a^s_{n+2}}{M^s_{n+3}} - \ldots\right] = 0\,. \quad (57).$$

In each case the former continued fraction terminates with a partial quotient involving a^s_s in the numerator and either M^s_s or N^s_s in the denominator, while the latter proceeds to infinity.

For the symmetrical types, if we use the form (56) we must suppose $n - s$ an even integer, whereas if we employ (57) $n - s$ must be supposed odd. The reverse will of course be the case for the asymmetrical types.

The continued fractions of the present section will not converge so rapidly as those of the preceding, but in spite of this drawback they present considerable advantages. In the first place the numerators of the partial quotients, which are obtained by giving n different integral values in the expression (54), are independent of λ. These numerators, which further are in a convenient form for logarithmic computation, may therefore be tabulated once for all, whereas the numerators of the partial quotients in the continued fractions of the last section require to be re-determined at each successive trial in attempting to solve the period-equation by trial and error. Moreover, the evaluation of the denominators M^s_n, N^s_n by means of the formulæ (51) may be very quickly effected, even though a fairly large number of these denominators is required, whereas the evaluation of the quantities L^s_n by means of (29) and (32) is extremely laborious.

Another disadvantage resulting from the use of the preceding form is that when λ/ω is near the value $2s/n\,(n + 1)$ the functions x^s_{n-1}, y^s_{n-1} both become large, while L^s_{n-1}, L^s_{n+1} have both a zero and an infinity in the immediate proximity of this value. Hence, in order to evaluate L^s_{n-1}, L^s_{n+1} for a value of λ in this region it is necessary to observe a very high degree of accuracy in the numerical work. The singularities which occur in the left-hand member of (46) when λ passes through one of these critical values no longer appear if we write the period-equation in the form (57) with the value of n appropriately chosen.

§ 7. *Expressions for the Velocity-Components.*

The auxiliary constants D^s_s, D^s_{s+1}, &c., introduced in the last section, may be made use of to express the velocity-components by means of series of surface-harmonics.

Thus from the first of equations (13) we have, on replacing ψ by its value in terms of Ψ_1, Ψ_2,

$$\sqrt{(1-\mu^2)}\,\mathrm{U} = -\frac{is\sigma}{2\omega a}[\Psi_1 + (\mathrm{D}-\sigma\mu)\Psi_2].$$

On introducing the expansions for Ψ_1, Ψ_2 on the right we find

$$\sqrt{(1-\mu^2)}\,\mathrm{U} = -\frac{is\sigma}{2\omega a}\Sigma\left[\alpha_n^s \mathrm{P}_n^s + \beta_n^s\left\{-\frac{(n+\sigma)(n-s+1)}{2n+1}\mathrm{P}_{n+1}^s + \frac{(n-\sigma+1)(n+s)}{2n+1}\mathrm{P}_{n-1}^s\right\}\right],$$

$$= -\frac{is\sigma}{2\omega a}\Sigma\mathrm{P}_n^s\left[\alpha_n^s - \frac{(n+\sigma-1)(n-s)}{2n-1}\beta_{n-1}^s + \frac{(n-\sigma+2)(n+s+1)}{2n+3}\beta_{n+1}^s\right],$$

which, with the help of (26), (27), reduces to

$$\sqrt{(1-\mu^2)}\,\mathrm{U} = \frac{2\omega isa}{\sigma h}\Sigma\mathrm{P}_n^s\left[-\frac{(n-s)(n+\sigma-1)}{(2n-1)\{n(n+1)-\sigma\}}\mathrm{C}_{n-1}^s + \frac{(n+s+1)(n-\sigma)}{(2n+3)\{n(n+1)-\sigma\}}\mathrm{C}_{n+1}^s\right],$$

or from (49),

$$\sqrt{(1-\mu^2)}\,\mathrm{U} = \frac{i\lambda a}{h}\Sigma\left\{\frac{n-s}{2n-1}\mathrm{C}_{n-1}^s + \frac{n+s+1}{2n+3}\mathrm{C}_{n+1}^s - \mathrm{D}_n^s\right\}\mathrm{P}_n^s \quad . \quad . \quad (58).$$

This may also be written in the form

$$\sqrt{(1-\mu^2)}\,\mathrm{U} = \frac{i\lambda a}{h}\Sigma\{\mu\mathrm{C}_n^s - \mathrm{D}_n^s\}\mathrm{P}_n^s \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).$$

Again, from (13)

$$\sqrt{(1-\mu^2)}\,\mathrm{V} = -\frac{i\sigma}{s}\mu\sqrt{(1-\mu^2)}\,\mathrm{U} + \frac{\sigma}{2\omega a}\psi$$

$$= +\frac{2\omega a}{h}\mu\Sigma(\mu\mathrm{C}_n^s - \mathrm{D}_n^s)\mathrm{P}_n^s + \frac{s}{a\lambda}\Sigma\Gamma_n^s\mathrm{P}_n^s \quad . \quad . \quad . \quad (60),$$

whence, by means of (6), we may express $\sqrt{(1-\mu^2)}\,\mathrm{V}$ by a series of surface-harmonics.

The corresponding formulæ when the depth is variable may be obtained by replacing h by κ and C_n^s by

$$\mathrm{C}_n^s + \frac{l}{4\omega^2a^2}\left\{n(n+1) + \frac{2\omega s}{\lambda}\right\}\Gamma_n^s,$$

so that we find

$$\left.\begin{aligned} \sqrt{(1-\mu^2)}\,\mathrm{U} &= \frac{i\lambda a}{\kappa}\Sigma\left(\mu\left[\mathrm{C}_n^s + \frac{l}{4\omega^2a^2}\left\{n(n+1)+\frac{2\omega s}{\lambda}\right\}\Gamma_n^s\right] - \mathrm{D}_n^s\right)\mathrm{P}_n^s \\ \sqrt{(1-\mu^2)}\,\mathrm{V} &= \frac{2\omega a\mu}{\kappa}\Sigma\left(\mu\left[\mathrm{C}_n^s + \frac{l}{4\omega^2a^2}\left\{n(n+1)+\frac{2\omega s}{\lambda}\right\}\Gamma_n^s\right] - \mathrm{D}_n^s\right)\mathrm{P}_n^s + \frac{s}{a\lambda}\Sigma\Gamma_n^s\mathrm{P}_n^s \end{aligned}\right\} \quad (61)$$

where the quantities D_n^s are now defined by

$$N_n^s D_n^s = \frac{(n+1)^2(n-s)}{2n-1}\left[C_{n-1}^s + \frac{l}{4\omega^2 a^2}\left\{n(n-1)+\frac{2\omega s}{\lambda}\right\}\Gamma_{n-1}^s\right]$$
$$+ \frac{n^2(n+s+1)}{2n+3}\left[C_{n+1}^s + \frac{l}{4\omega^2 a^2}\left\{(n+1)(n+2)+\frac{2\omega s}{\lambda}\right\}\Gamma_{n+1}^s\right] \quad (62).$$

The formulæ (61) cease to be of use in a special case which will present itself hereafter for which $\kappa = 0$. It will be seen in a later section that the expressions on the right become indeterminate in this case, so that the determination of the velocity-components must be effected by means of the formulæ (12) or (13). These latter formulæ seem at first sight to indicate that the velocity-components become infinite in latitude $\sin^{-1}(s/\sigma)$, but the forms (61) indicate that such cannot be the case, at least when κ is different from zero.

§ 8. *Approximate Determination of the Higher Roots of the Period-Equation.*

If we take the period-equation in the form (56), and as a first approximation omit the continued fractions from the left-hand member, it reduces to the quadratic

$$M_n^s = 0,$$

or

$$\frac{\lambda^2}{4\omega^2}\left\{n(n+1)-\frac{2\omega s}{\lambda}\right\} - n^2(n+1)^2\frac{hg_n}{4\omega^2 a^2} = 0.$$

For large values of n the roots of this equation will give a sufficiently accurate approximation to the roots of the period-equation, since it may be seen that the continued fractions tend to limits comparable with $\frac{1}{4}n^2$, and therefore small in comparison with $n^2(n+1)^2\frac{hg_n}{4\omega^2a^2}$, when n is very large and λ/ω has as its value either of the roots of this equation. We may even obtain a fair approximation by omitting the term containing s, in which case the formula for λ corresponds with that obtained when the rotation is omitted.

A better approximation will however be obtained by representing the continued fractions by their first convergents instead of entirely neglecting them. The approximate form of the period-equation is then

$$M_n^s - \frac{a_{n-1}^s}{N_{n-1}^s} - \frac{a_n^s}{N_{n+1}^s} = 0,$$

or

$$\frac{\lambda^2}{4\omega^2}\,\frac{\{n(n+1)-2\omega s/\lambda\}}{n^2(n+1)^2} - \frac{(n-1)^2(n-s)(n+s)}{n^2(2n-1)(2n+1)\{(n-1)n-2\omega s/\lambda\}}$$
$$- \frac{(n+2)^2(n-s+1)(n+s+1)}{(n+1)^2(2n+1)(2n+3)\{(n+1)(n+2)-2\omega s/\lambda\}} - \frac{hg_n}{4\omega^2a^2} = 0.$$

We thus get back to the equation $L_n^s = 0$. The roots of this equation may be approximated to numerically by HORNER's process, the significant roots being those which lie in the intervals between $-\infty$ and 0, and between $2s/n(n-1)$ and $+\infty$.

For the particular case $s = 0$, the biquadratic to which the equation $L^s_n = 0$ is equivalent reduces to a quadratic, the two roots which remain finite being of equal magnitude and opposite sign. This special case has been examined in Part I., and it will be seen on reference to the tables there given (§§ 7–8), that the roots of the equation $L_n = 0$ give a very good approximation to the roots of the period-equation except in the case of the earlier roots when $hg/4\omega^2a^2$ is small. In the present paper I have examined in some detail the special cases corresponding to the values 1 and 2 for s, and the approximation is found to be equally rapid, as will be seen from the tables given hereafter. Consequently, in these cases at least, all except the two or three smallest roots will be obtained with adequate accuracy by finding the roots which lie in the stated intervals of the equations $L^s_n = 0$ with different integral values of n.

The roots so found will not however form the complete series of roots of the period-equation. We may in fact anticipate that the remaining roots of the equation $L^s_n = 0$ will also approximate to roots of the period-equation. To obtain a better approximation of the roots of this class, it will however be preferable to make use of the period-equation in the form (57). As a first approximation we omit the continued fractions and obtain

$$N^s_n = 0 \quad \text{or} \quad n(n+1) - \frac{2\omega s}{\lambda} = 0.$$

This method of approximation will be valid if when $\lambda/\omega = 2s/n(n+1)$ the two continued fractions involved in (57) are small in comparison with $n(n+1)$. But it may readily be verified that with large values of n these continued fractions become comparable with ω^2a^2/hg, and therefore the desired condition will certainly be satisfied when n is sufficiently large.

A better approximation may be obtained by representing the continued fractions by their first convergents. We thus obtain as the approximate form of the period-equation for the determination of the root which lies near $\dfrac{2s}{n(n+1)}$

$$N^s_n - \frac{a^s_{n-1}}{M^s_{n-1}} - \frac{a^s_n}{M^s_{n+1}} = 0,$$

or

$$n(n+1) - \frac{2\omega s}{\lambda} = \frac{\dfrac{(n-1)^2(n+1)^2(n-s)(n+s)}{(2n-1)(2n+1)}}{\dfrac{\lambda^2}{4\omega^2}\left\{n(n-1) - \dfrac{2\omega s}{\lambda}\right\} - \dfrac{n^2(n-1)^2hg_{n-1}}{4\omega^2a^2}}$$
$$+ \frac{\dfrac{n^2(n+2)^2(n-s+1)(n+s+1)}{(2n+1)(2n+3)}}{\dfrac{\lambda^2}{4\omega^2}\left\{(n+1)(n+2) - \dfrac{2\omega s}{\lambda}\right\} - \dfrac{(n+1)^2(n+2)^2hg_{n+1}}{4\omega^2a^2}}.$$

Using the first approximation in the terms on the right, we deduce

$$n(n+1)-\frac{2\omega s}{\lambda}=-\frac{\dfrac{(n-1)^2(n+1)^2(n-s)(n+s)}{(2n-1)(2n+1)}}{n^2(n-1)^2\dfrac{hg_{n-1}}{4\omega^2a^2}+\dfrac{2s^2}{n(n+1)^2}}-\frac{\dfrac{n^2(n+2)^2(n-s+1)(n+s+1)}{(2n+1)(2n+3)}}{(n+1)^2(n+2)^2\dfrac{hg_{n+1}}{4\omega^2a^2}+\dfrac{2s^2}{n^2(n+1)}} \quad (63).$$

This formula is found to lead to the roots of the period-equation with a surprising degree of accuracy.

Our analysis is only applicable when h is small in comparison with a, but subject to this limitation the approximations of the present section will improve as $hg/4\omega^2a^2$ increases, that is, as the depth of the water increases or the angular velocity of rotation diminishes. They will give good results even with small values of n when ω is sufficiently small, and they may be used to determine the limiting values assumed by the roots when the angular velocity of rotation is indefinitely reduced.

We see then that the roots of the period-equation are of two classes, which may be distinguished by their limiting forms when the rotation is annulled. The roots of the former class are such that the values of λ remain finite when $\omega = 0$, their limiting values being given by the formula

$$\lambda = \pm\sqrt{\frac{n(n+1)hg_n}{a^2}}.$$

There will be an equal number of positive and negative roots of this class, but though these approach the same limiting values their numerical values will not be equal as in the case where $s = 0$, and the positive and negative roots must therefore be determined independently.

The roots of the second class are all positive and are such that the values of λ/ω tend to finite limits when ω is reduced to zero, the limiting values being given by the formula

$$\frac{\lambda}{\omega} = \frac{2s}{n(n+1)},$$

whereas λ will tend to the limit zero.

The analogue of the types of motion which correspond with the former roots will still be oscillatory when the rotation is annulled, but the types of motion corresponding with the roots of the second class will cease to exist as oscillations when the angular velocity of rotation is reduced to zero. These types of motion will have their equivalent in steady motions, but an infinitesimal amount of rotation would immediately convert such steady motions into oscillatory motions of very long period.

For the particular case $s = 0$ the roots of the second class are all zero even when the angular velocity of rotation is finite. Hence steady motions can exist on a rotating globe, but these are necessarily of zonal type. We have in fact already seen in Part I.* that the only forms of steady motion which can exist are of this character, and have explained the fact by stating that the steady motions not of zonal type which can exist on a globe without rotation must have their analogue in the more general case in oscillatory motions whose period bears a finite ratio to the rotation-period, no matter how great the latter may be. Our present work confirms this statement and throws further light on the nature of these oscillatory motions.

§ 9. *Evaluation of the Earlier Roots.*

The errors resulting from the use of the approximate formulæ of the last section may be considerable in the case of the earlier roots for which n has small values. To obtain these earlier roots we must therefore proceed by trial and error, the preceding method being made use of to obtain values with which to commence the trials.

As a concrete example we will discuss in detail the computation of the positive root of the first class corresponding to the case $n = 4$, $s = 1$, when the depth is given by $hg/4\omega^2a^2 = \frac{1}{20}$. Taking $\rho/\sigma_0 = 0·18093$, and introducing the numerical values of n, s, and h, the equation $L_4^1 = 0$ becomes

$$(\lambda/\omega)^4 - 0·3333\,(\lambda/\omega)^3 - 5·5481\,(\lambda/\omega)^2 + 1·0906\,(\lambda/\omega) - 0·0418 = 0.$$

By Horner's process the greatest positive root of this equation is found to be

$$2·43265.$$

Now experience shows that the numerical value thus suggested is in general too small.† We therefore select for a first trial a value rather larger than that indicated, say, for example,

$$\lambda/\omega = 2·4400.$$

From the formula (54) we find

$$\begin{aligned} \log a_1^1 &= 0·2553, & \log a_2^1 &= 1·1652, & \log a_3^1 &= 1·7289,\\ \log a_4^1 &= 2·1450, & \log a_5^1 &= 2·4769, & \log a_6^1 &= 2·7537,\\ \log a_7^1 &= 2·9915, & \log a_8^1 &= 3·2001, & \log a_9^1 &= 3·3859, \end{aligned}$$

while the values of the expression

$$n^2(n+1)^2\,\frac{hg_n}{4\omega^2a^2}$$

for the values 2, 4, 6, 8, 10 of n are

$$1·6046, \quad 18·794, \quad 84·517, \quad 250·92, \quad 589·4.$$

* §§ 14, 15.

† Compare the 2nd and 3rd columns of Tables I. and II., Part I.

Thus we find from the formulæ (51), with $\lambda/\omega = 2{\cdot}4400$,

$$M_2^1 = 6{\cdot}104, \quad M_4^1 = 9{\cdot}753, \quad M_6^1 = -23{\cdot}25, \quad M_8^1 = -144{\cdot}97, \quad M_{10}^1 = -426{\cdot}9,$$
$$N_1^1 = 1{\cdot}180, \quad N_3^1 = 11{\cdot}180, \quad N_5^1 = 29{\cdot}180, \quad N_7^1 = 55{\cdot}180, \quad N_9^1 = 89{\cdot}18.$$

It will be convenient for us now to introduce the following abridged notation :—

$$\left.\begin{aligned} e_n^s &= \frac{a_{n-1}^s}{M_n^s -} \frac{a_n^s}{N_{n+1}^s -} \frac{a_{n+1}^s}{M_{n+2}^s -} \dots \textit{ad inf.} \\ f_n^s &= \frac{a_{n-1}^s}{N_n^s -} \frac{a_n^s}{M_{n+1}^s -} \frac{a_{n+1}^s}{N_{n+2}^s -} \dots \textit{ad inf.} \\ E_n^s &= \frac{a_n^s}{M_n^s -} \frac{a_{n-1}^s}{N_{n-1}^s -} \frac{a_{n-2}^s}{M_{n-2}^s -} \dots \\ F_n^s &= \frac{a_n^s}{N_n^s -} \frac{a_{n-1}^s}{M_{n-1}^s -} \frac{a_{n-2}^s}{N_{n-2}^s -} \dots \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (64),$$

the last two continued fractions terminating with the partial quotient which involves a_s^s in the numerator, and either M_s^s or N_s^s in the denominator.

From these definitions of the quantities e, f, E, F, we have :

$$\left.\begin{aligned} e_n^s &= \frac{a_{n-1}^s}{M_n^s - f_{n+1}^s}, & f_n^s &= \frac{a_{n-1}^s}{N_n^s - e_{n+1}^s} \\ E_n^s &= \frac{a_n^s}{M_n^s - F_{n-1}^s}, & F_n^s &= \frac{a_n^s}{N_n^s - E_{n-1}^s} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (65);$$

while the period-equation may be written in the forms :

$$\left.\begin{aligned} M_n^s - F_{n-1}^s - f_{n+1}^s = 0 \\ N_n^s - E_{n-1}^s - e_{n+1}^s = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66).$$

Suppose that we neglect f_{11}^1; making use of the numerical values obtained above for the quantities M, N, a, by successive applications of the formulæ (65) we obtain

$$\log e_{10}^1 = n0{\cdot}7556, \quad \log f_9^1 = 1{\cdot}2229, \quad \log e_8^1 = n0{\cdot}7829,$$
$$\log f_7^1 = 0{\cdot}9666, \quad \log e_6^1 = n0{\cdot}9649, \quad \log f_5^1 = 0{\cdot}5606.$$

In like manner, if we neglect e_{10}^1, we find

$$\log f_9^1 = 1{\cdot}2498, \quad \log e_8^1 = n0{\cdot}7801, \quad \log f_7^1 = 0{\cdot}9668,$$
$$\log e_6^1 = n0{\cdot}9649, \quad \log f_5^1 = 0{\cdot}5606.$$

Now the two values of f_5^1 obtained by these methods are respectively the 6th and

5th convergents of the continued fraction f_5^1. Since we find that to the degree of accuracy retained these are equal, it follows that all subsequent convergents are sensibly equal to either of them. Hence the infinite continued fraction f_5^1 may be replaced by its fifth convergent without sensible error.

Similarly we find

$$\log F_1^1 = 0{\cdot}1834, \quad \log E_2^1 = 0{\cdot}5045, \quad \log F_3^1 = 0{\cdot}8266,$$

and therefore

$$M_4^1 - F_3^1 - f_5^1 = 9{\cdot}753 - 6{\cdot}707 - 3{\cdot}636 = -0{\cdot}590.$$

As a second trial we take

$$\lambda/\omega = 2{\cdot}4600.$$

Proceeding as before, we deduce

$$M_4^1 - F_3^1 - f_5^1 = 10{\cdot}234 - 6{\cdot}619 - 3{\cdot}607 = 0{\cdot}008.$$

We conclude that there is a root of the period-equation lying between 2·4400 and 2·4600 ; by interpolation its value is found to be

$$2{\cdot}4597.$$

The same method may be used for the determination of the roots of the second class, the initial trial values being suggested by the formula (63). As a numerical example, if we put $n = 5$, $s = 1$, $hg/4\omega^2a^2 = \frac{1}{20}$ in (63), we find

$$2\omega/\lambda = 40{\cdot}974, \quad \text{or} \quad \lambda/\omega = 0{\cdot}04881.$$

For a first trial we take $2\omega/\lambda = 41$, and deduce

$$N_5^1 - E_4^1 - e_6^1 = -11 + 8{\cdot}678 + 2{\cdot}602 = 0{\cdot}280.$$

As a second trial we take $2\omega/\lambda = 41{\cdot}280$, and obtain

$$N_5^1 - E_4^1 - e_6^1 = -11{\cdot}280 + 8{\cdot}657 + 2{\cdot}593 = -0{\cdot}030,$$

and therefore, by interpolation,

$$N_5^1 - E_4^1 - e_6^1 = 0,$$

when

$$2\omega/\lambda = 41{\cdot}253, \quad \text{or} \quad \lambda/\omega = 0{\cdot}04848.$$

I have selected for special investigation the asymmetrical types when $s = 1$, and the symmetrical types when $s = 2$, these types presenting special interest in relation to the diurnal and semi-diurnal forced tides. The annexed tables give the values of the computed roots, together with the corresponding periods of free oscillation for the four depths of 7260, 14,520, 29,040, 58,080 feet treated of in Part I., corresponding to the values $\frac{1}{40}$, $\frac{1}{20}$, $\frac{1}{10}$, and $\frac{1}{5}$ for $hg/4\omega^2a^2$.

I.—DEPTH 7260 feet ($hg/4\omega^2a^2 = \frac{1}{40}$, $\rho/\sigma_0 = 0{\cdot}18093$).

		s = 1.			s = 2.		
		Approximate root.	Corrected root.	Period.*	Approximate root.	Corrected root.	Period.
Class I.				hrs. mins.			hrs. mins.
	$n = 2$	1·8331	1·6337	14 41	1·4745	1·3347	17 59
		− 1·0906	− 0·9834	24 24	− 0·6582	− 0·6221	38 34
	$n = 4$	2·0042	2·0685	11 36	2·0024	1·9866	12 5
		− 1·8472	− 1·8234	13 10	− 1·7006	− 1·6595	14 28
Class II.				days hrs.			days hrs.
	$n = 1$	0·6932	0·6401	1 13	..	..	..
	$n = 3$	0·05702	0·06251	16 0	0·1532	0·1671	6 0
	$n = 5$	0·03853	0·03784	26 10	0·08091	0·08178	12 5

II.—DEPTH 14,520 feet ($hg/4\omega^2a^2 = \frac{1}{20}$, $\rho/\sigma_0 = 0{\cdot}18093$).

		s = 1.			s = 2.		
		Approximate root.	Corrected root.	Period.	Approximate root.	Corrected root.	Period.
Class I.				hrs. mins.			hrs. mins.
	$n = 2$	1·9588	1·8677	12 51	1·6804	1·6133	14 52
		− 1·3021	− 1·2450	19 16	− 0·9128	− 0·8922	26 54
	$n = 4$	2·4327	2·4597	9 45	2·4363	2·4349	9 51
		− 2·2949	− 2·2907	10 29	− 2·1706	− 2·1547	11 8
Class II.				days hrs.			days hrs.
	$n = 1$	0·7502	0·7283	1 9	..	..	..
	$n = 3$	0·08423	0·08673	11 13	0·2054	0·2129	4 17
	$n = 5$	0·04881	0·04848	20 15	0·10067	0·10081	9 22

* Throughout these tables the periods are expressed in sidereal time.

III.—Depth 29,040 feet ($hg/4\omega^2a^2 = \frac{1}{10}$, $\rho/\sigma_0 = 0{\cdot}18093$).

		$s = 1$.			$s = 2$.		
		Approximate root.	Corrected root.	Period.	Approximate root.	Corrected root.	Period.
				hrs. mins.			hrs. mins.
Class I.	$n = 2$	2·2027	2·1641	11 5	2·0241	1·9968	12 1
		− 1·6439	− 1·6170	14 50	− 1·2960	− 1·2855	18 40
	$n = 4$	3·1183	3·1274	7 40	3·1295	3·1293	7 40
		− 2·9958	− 2·9961	8 1	− 2·8911	− 2·8856	8 19
				days hrs.			days hrs.
Class II.	$n = 1$	0·8213	0·8149	1 5	..	..	..
	$n = 3$	0·1116	0·1124	8 21	0·2523	0·2554	3 22
	$n = 5$	0·05636	0·05625	17 19	0·11472	0·11472	8 17

IV.—Depth 58,080 feet ($hg/4\omega^2a^2 = \frac{1}{5}$, $\rho/\sigma_0 = 0{\cdot}18093$).

		$s = 1$.			$s = 2$.		
		Approximate root.	Corrected root.	Period.	Approximate root.	Corrected root.	Period.
				hrs. mins.			hrs. mins.
Class I.	$n = 2$	2·6431	2·6288	9 8	2·5636	2·5535	9 24
		− 2·1726	− 2·1611	11 6	− 1·8623	− 1·8575	12 55
	$n = 4$	4·1626	4·1659	5 46	4·1820	4·1818	5 44
		− 4·0501	− 4·0506	5 56	− 3·9610	− 3·9592	6 4
				days hrs.			days hrs.
Class II.	$n = 1$	0·8886	0·8873	1 3	..	..	..
	$n = 3$	0·13354	0·13375	7 11	0·2865	0·2876	3 11
	$n = 5$	0·06108	0·06105	16 9	0·12332	0·12332	8 3

The approximate roots here given have been evaluated by the method of § 7, except in the case of the roots of the second class for $n = 1$, $s = 1$, where, instead of replacing the continued fraction involved in the period-equation by its first convergent, I have made use of the second convergent, so that the approximate form of the period-equation from which this root is determined is

$$N_1^1 - \frac{\alpha_1^1}{M_2^1} - \frac{\alpha_2^1}{N_3^1} = 0\,,$$

or

$$M_2^1 - \frac{\alpha_1^1}{N_1^1} - \frac{\alpha_2^1}{N_3^1} = 0\,,$$

or, what is equivalent, $L_2^1 = 0$.

By a comparison of the approximate values given in these tables with the true values we see that for extreme cases here tabulated, the error involved in the approximation does not amount to more than about 3 per cent., even with a depth as small as 7260 feet. We have here a justification of the statements made in the last section as to the approximation to the higher roots.

§ 10. *Oscillations of the First Class. Determination of the Type.*

We shall describe as oscillations of the first class those whose periods remain finite when the rotation-period is indefinitely prolonged, that is, those for which the roots of the period-equation are of the first class. The types of motion whose periods become infinitely long with the rotation-period will be called oscillations of the second class.

The determination of the type involves the determination of the constants C^s_s, D^s_{s+1}, C^s_{s+2}, &c. (supposing for convenience that we are dealing with symmetrical types). For this purpose we may either make use of the formulæ of § 6, or we may make use of the formulæ of § 5 for the determination of the C's, after which the D's must be computed from equation (49). The latter method will be closely analogous to that used in § 9 of Part I., but the former is the more convenient when the numerical determination of the constants is required.

If we make use of the notation (64), the following relations may be deduced from the equations (52) :—

$$\left.\begin{aligned} \frac{C^s_n}{D^s_{n-1}} &= \frac{2n+1}{(n+s)(n-1)^2}\, e^s_n, & \frac{C^s_n}{D^s_{n+1}} &= \frac{2n+1}{(n-s+1)(n+2)^2}\, E^s_n \\ \frac{D^s_n}{C^s_{n-1}} &= \frac{2n+1}{(n+s)(n-1)^2}\, f^s_n, & \frac{D^s_n}{C^s_{n+1}} &= \frac{2n+1}{(n-s+1)(n+2)^2}\, F^s_n \end{aligned}\right\} \quad . \quad . \quad (67).$$

But we have seen in the last section how the quantities e, f, E, F may be determined numerically. Hence the above formulæ allow us to compute the ratios of the constants C, D. One of these constants must be regarded as arbitrary, and the ratios of the others to it can then be computed. When the type under examination is that which corresponds to a root of the period-equation approximating to a root of $M^s_n = 0$, we select as the arbitrary constant of integration the quantity C^s_n, as the continued fractions e, f, E, F required to determine the ratios of the remaining constants to this one will then be free from singularities.

When these ratios have been determined we may substitute their values in the formulæ

$$\zeta = C^s_n e^{i(\lambda t + s\phi)} \left[\ldots + \frac{C^s_{n-4}}{C^s_n} P^s_{n-4} + \frac{C^s_{n-2}}{C^s_n} P^s_{n-2} + P^s_n + \frac{C^s_{n+2}}{C^s_n} P^s_{n+2} + \frac{C^s_{n+4}}{C^s_n} P^s_{n+} + \ldots \right]$$

$$\sqrt{(1-\mu^2)}\,\mathrm{U} = \frac{i\lambda a}{h}\,\mathrm{C}_n^s e^{i(\lambda t + s\phi)}\left[\mu\left\{\ldots + \frac{\mathrm{C}_{n-2}^s}{\mathrm{C}_n^s}\mathrm{P}_{n-2}^s + \mathrm{P}_n^s + \frac{\mathrm{C}_{n+2}^s}{\mathrm{C}_n^s}\mathrm{P}_{n+2}^s + \ldots\right\}\right.$$
$$\left. - \left\{\ldots + \frac{\mathrm{D}_{n-1}^s}{\mathrm{C}_n^s}\mathrm{P}_{n-1}^s + \frac{\mathrm{D}_{n+1}^s}{\mathrm{C}_n^s}\mathrm{P}_{n+1}^s + \ldots\right\}\right]$$

$$\sqrt{(1-\mu^2)}\,\mathrm{V} = \frac{2\omega a}{h}\,\mu\,\mathrm{C}_n^s e^{i(\lambda t + s\phi)}\left[\mu\left\{\ldots + \frac{\mathrm{C}_{n-2}^s}{\mathrm{C}_n^s}\mathrm{P}_{n-2}^s + \mathrm{P}_n^s + \frac{\mathrm{C}_{n+2}^s}{\mathrm{C}_n^s}\mathrm{P}_{n+2}^s + \ldots\right\}\right.$$
$$\left. - \left\{\ldots + \frac{\mathrm{D}_{n-1}^s}{\mathrm{C}_n^s}\mathrm{P}_{n-1}^s + \frac{\mathrm{D}_{n+1}^s}{\mathrm{C}_n^s}\mathrm{P}_{n+1}^s + \ldots\right\}\right]$$
$$- \frac{s}{a\lambda}\,\mathrm{C}_n^s e^{i(\lambda t + s\phi)}\left[\ldots + \frac{\mathrm{C}_{n-2}^s}{\mathrm{C}_n^s}g_{n-2}\,\mathrm{P}_{n-2}^s + g_n\,\mathrm{P}_n^s + \frac{\mathrm{C}_{n+2}^s}{\mathrm{C}_n^s}g_{n+2}\,\mathrm{P}_{n+2}^s + \ldots\right]$$

and we shall obtain expressions for the height of the surface-waves and the velocity-components.

I have not thought it worth while to compute any of these series in detail, as the general character of them may be inferred from the series computed in Part I. for the special case where $s = 0$. For large values of n, and even for comparatively small values of n when $hg/4\omega^2a^2$ is large, the quantities C will rapidly diminish as we pass away in either direction from C_n^s. Hence the most important term in the series for ζ will be that involving P_n^s, and this term will in general sufficiently predominate to decide the number and approximate position of the nodal parallels of latitude.

If we neglect C_{r-1}^s in comparison with C_{r+1}^s and suppose that ω is small in comparison with λ, the formula (49) gives

$$r\,(r + s + 1)\,\mathrm{C}_{r+1}^s = (r + 1)\,(2r + 3)\,\mathrm{D}_r^s$$

and therefore D_r^s will be of the same order of magnitude as C_{r+1}^s. Hence, when r is less than n, D_r^s will be of the same order of magnitude of C_{r+1}^s, and similarly it may be seen that when r is greater than n, D_r^s will be of the same order of magnitude as C_{r-1}^s. Thus the predominant terms in the expressions for the velocity-components will be those involving C_n^s, D_{n-1}^s, D_{n+1}^s.

The exponential factors indicate that the type of motion involved will consist of waves propagated round the sphere with uniform angular velocity λ/s about the polar axis, and that there will be s crests or troughs on each parallel. Positive values of λ will correspond with waves propagated in the opposite direction to the rotation, that is westwards, while negative values will correspond with easterly waves. The paths of the fluid particles will be ellipses with their axes directed along the meridians and parallels.

§ 11. *Oscillations of the Second Class.*

In dealing with the oscillations of the second class we proceed in the same manner as before, retaining D_n^s as the arbitrary constant of integration when the type under consideration is that whose period is approximately given by the formula

$$n(n+1) - 2\omega s/\lambda = 0.$$

For we may anticipate that this quantity will predominate over the others, at least when the depth is sufficiently large or the angular velocity of rotation sufficiently small. The ratios of the remaining constants to D_n^s may then be computed from the formulæ (67), and on substituting these ratios in the equations

$$\zeta = D_n^s e^{i(\lambda t + s\phi)}\left[\ldots + \frac{C_{n-3}^s}{D_n^s} P_{n-3}^s + \frac{C_{n-1}^s}{D_n^s} P_{n-1}^s + \frac{C_{n+1}^s}{D_n^s} P_{n+1}^s + \frac{C_{n+3}^s}{D_n^s} P_{n+3}^s + \ldots\right]$$

$$\sqrt{(1-\mu^2)}\,U = \frac{i\lambda a}{h} D_n^s e^{i(\lambda t + s\phi)}\left[\mu\left\{\ldots + \frac{C_{n-1}^s}{D_n^s} P_{n-1}^s + \frac{C_{n+1}^s}{D_n^s} P_{n+1}^s + \ldots\right\} - \left\{\ldots + \frac{D_{n-2}^s}{D_n^s} P_{n-2}^s + P_n^s + \frac{D_{n+2}^s}{D_n^s} P_{n+2}^s + \ldots\right\}\right]$$

$$\sqrt{(1-\mu^2)}\,V = \frac{2\omega a\mu}{h} D_n^s e^{i(\lambda t + s\phi)}\left[\mu\left\{\ldots + \frac{C_{n-1}^s}{D_n^s} P_{n-1}^s + \frac{C_{n+1}^s}{D_n^s} P_{n+1}^s + \ldots\right\} - \left\{\ldots + \frac{D_{n-2}^s}{D_n^s} P_{n-2}^s + P_n^s + \frac{D_{n+2}^s}{D_n^s} P_{n+2}^s + \ldots\right\}\right]$$
$$- \frac{s}{a\lambda} D_n^s e^{i(\lambda t + s\phi)}\left[\ldots + \frac{C_{n-1}^s}{D_n^s} g_{n-1} P_{n-1}^s + \frac{C_{n+1}^s}{D_n^s} g_{n+1} P_{n+1}^s + \ldots\right],$$

we obtain expressions for the height of the surface-waves and the velocity-components. The values of λ for the oscillations of this class being always positive, the direction of the wave-propagation will always be westwards.

By way of numerical illustration I have computed the series for ζ corresponding to the case $n = 3$, $s = 1$, and to the values $\frac{1}{40}$, $\frac{1}{20}$, $\frac{1}{10}$, $\frac{1}{5}$ for $hg/4\omega^2 a^2$. In these four cases the series within the square brackets are found to be

$$- 2{\cdot}977 P_2^1 - 0{\cdot}1880 P_4^1 + 0{\cdot}3753 P_6^1 - 0{\cdot}0916 P_8^1 + 0{\cdot}01153 P_{10}^1 - 0{\cdot}00093 P_{12}^1 + 0{\cdot}000052 P_{14}^1 - 0{\cdot}000002 P_{16}^1 + \ldots$$

$$- 1{\cdot}4735 P_2^1 - 0{\cdot}3260 P_4^1 + 0{\cdot}11690 P_6^1 - 0{\cdot}01309 P_8^1 + 0{\cdot}00080 P_{10}^1 - 0{\cdot}000032 P_{12}^1 + 0{\cdot}000001 P_{14}^1 - \ldots$$

$$- 0{\cdot}7296 P_2^1 - 0{\cdot}2248 P_4^1 + 0{\cdot}03159 P_6^1 - 0{\cdot}00168 P_8^1 + 0{\cdot}000051 P_{10}^1 - 0{\cdot}000001 P_{12}^1 + \ldots$$

$$- 0{\cdot}3617 P_2^1 - 0{\cdot}1277 P_4^1 + 0{\cdot}00814 P_6^1 - 0{\cdot}00021 P_8^1 + {\cdot}000003 P_{10}^1 - \ldots$$

It will be seen that as the depth increases or ω diminishes the coefficients all become smaller, while the convergence of the series improves. It may be inferred as in the last section that C_r^s/D_{r+1}^s will tend towards the limit zero when r is less than

n, while C^s_{r+1}/D^s_r will tend towards a finite limit. In like manner when r is greater than n, C^s_r/D^s_{r-1} will tend towards zero while C^s_{r-1}/D^s_r will tend to a finite limit.

Let us examine the limiting forms assumed by ζ, U, V, when the rotation is annulled. On putting $n(n+1) - 2\omega s/\lambda = 0$ the relations (51) give

$$\mathrm{L}^t\, \omega^2 \mathrm{M}^s_r = -r^2(r+1)^2 \frac{hg_r}{4a^2},$$

$$\mathrm{L}^t\, \mathrm{N}^s_r = r(r+1) - n(n+1) = (r-n)(r+n+1).$$

Thus from (52) we obtain, if $r < n$,

$$\mathrm{L}^t \frac{\mathrm{C}^s_r}{\omega^2 \mathrm{D}^s_{r+1}} = -\frac{4a^2}{g_r h} \frac{(r+s+1)}{(r+1)^2(2r+3)}, \quad \mathrm{L}^t \frac{\mathrm{C}^s_{r+1}}{\mathrm{D}^s_r} = \frac{(r-n)(r+n+1)(2r+3)}{r^2(r+s+1)},$$

and if $r > n$,

$$\mathrm{L}^t \frac{\mathrm{C}^s_r}{\omega^2 \mathrm{D}^s_{r-1}} = -\frac{4a^2}{g_r h} \frac{(r-s)}{r^2(2r-1)}, \quad \mathrm{L}^t \frac{\mathrm{C}^s_{r-1}}{\mathrm{D}^s_r} = \frac{(r-n)(r+n+1)(2r-1)}{(r+1)^2(r-s)}.$$

Hence if we retain only the most significant terms and put $\omega \mathrm{D}^s_n = \Delta^s_n$, we find

$$\zeta = -\omega\, \Delta^s_n e^{i(\lambda t + s\phi)} \left[\frac{4a^2}{hg_{n-1}} \frac{(n+s)}{n^2(2n+1)} \mathrm{P}^s_{n-1} + \frac{4a^2}{hg_{n+1}} \frac{(n-s+1)}{(n+1)^2(2n+1)} \mathrm{P}^s_{n+1}\right]$$

$$\sqrt{(1-\mu^2)}\, \mathrm{U} = -\frac{2isa}{n(n+1)h} \Delta^s_n e^{i(\lambda t + s\phi)} \mathrm{P}^s_n$$

$$\sqrt{(1-\mu^2)}\mathrm{V} = -\frac{2a}{h} \mu \Delta^s_n e^{i(\lambda t + s\phi)} \mathrm{P}^s_n$$
$$+ \frac{n(n+1)}{2a} \Delta^s_n e^{i(\lambda t + s\phi)} \left[\frac{4a^2}{h} \frac{n+s}{n^2(2n+1)} \mathrm{P}^s_{n-1} + \frac{4a^2}{h} \frac{(n-s+1)}{(n+1)^2(2n+1)} \mathrm{P}^s_{n+1}\right].$$

If therefore we suppose that ω reduces to zero while Δ^s_n remains finite, ζ will reduce to zero, but the velocity-components will tend to finite limits given by

$$\sqrt{(1-\mu^2)}\, \mathrm{U} = -\frac{2isa}{n(n+1)h} \Delta^s_n \mathrm{P}^s_n(\mu)\, e^{is\phi},$$

$$\sqrt{(1-\mu^2)}\, \mathrm{V} = -\frac{2a}{h} \Delta^s_n \left[\frac{n-s+1}{(n+1)(2n+1)} \mathrm{P}^s_{n+1} - \frac{(n+s)}{n(2n+1)} \mathrm{P}^s_{n-1}\right] e^{is\phi}$$
$$= \frac{2a}{n(n+1)h} \Delta^s_n (1-\mu^2) \frac{d\mathrm{P}^s_n}{d\mu} e^{is\phi}$$

in virtue of (7).

Hence the steady motions to which the oscillations of the second class reduce when the rotation is annulled involve no deformation of the free surface. This of course may readily be verified by a direct method. For if u, v, w denote the velocity-components referred to fixed rectangular axes, it may be seen that all the conditions of the problem will be satisfied by

$$u = y\frac{\partial\chi}{\partial z} - z\frac{\partial\chi}{\partial y}, \quad v = z\frac{\partial\chi}{\partial x} - x\frac{\partial\chi}{\partial z}, \quad w = x\frac{\partial\chi}{\partial y} - y\frac{\partial\chi}{\partial x},$$

where χ is an arbitrary function of x, y, z independent of t. These solutions make $ux + vy + wz = 0$, and therefore involve no deformation of the surface. If we refer them to polar co-ordinates, they are equivalent to

$$U = \frac{1}{\sin\theta}\frac{\partial\chi}{\partial\phi}, \quad V = \frac{\partial\chi}{\partial\theta}, \quad W = 0,$$

or

$$\sqrt{(1-\mu^2)}\,U = \frac{\partial\chi}{\partial\phi}, \quad \sqrt{(1-\mu^2)}\,V = -(1-\mu^2)\frac{\partial\chi}{\partial\mu}.$$

This solution becomes identical with that found above if

$$\chi = -\frac{2a}{n(n+1)h}\Delta_n^s P_n^s(\mu)\,e^{is\phi}.$$

For the special case $s = 0$, the values of λ corresponding with the roots of the second class will be zero, and the corresponding types of motion steady, even when ω is finite. The types of steady motion will however involve a deformation of the free surface in all cases where the angular velocity of rotation is different from zero. These cases have been fully discussed in Part I., § 14.

§ 12. *Forced Oscillations. General Analytical Solution.*

The problem of the forced oscillations involves the determination of the quantities C_n^s in terms of γ_n^s from the equation (31) or (40). Dealing first with the case of uniform depth, and supposing that the disturbing potential involves only a single term $\gamma_n^s P_n^s(\mu)\,e^{i(\lambda t + s\phi)}$, we have to solve the simultaneous equations

$$\begin{aligned}
-L_s^s C_s^s + y_s^s C_{s+2}^s &= 0,\\
x_s^s C_s^s - L_{s+2}^s C_{s+2}^s + y_{s+2}^s C_{s+4}^s &= 0,\\
&\dots\dots\dots\\
x_{n-2}^s C_{n-2}^s - L_n^s C_n^s + y_n^s C_{n+2}^s &= h\gamma_n^s/4\omega^2 a^2,\\
&\dots\dots\dots
\end{aligned}$$

with the condition that $\underset{r=\infty}{L^t}\, C_r^s = 0$.

From these we deduce, as in § 5, that

$$\frac{C_{r-2}^s}{C_r^s} = \frac{H_{r-2}^s}{x_{r-2}^s}\,(r < n+1),$$

$$\frac{C_{r+}^s}{C_r^s} = \frac{K_{r+2}^s}{y_r^s}\,(r > n-1).$$

Thus the equation which involves γ_n^s becomes

$$C_n^s (H_{n-2}^s - L_n^s + K_{n+2}^s) = h\gamma_n^s/4\omega^2 a^2 ;$$

whence

$$C_n^s = \frac{h\gamma_n^s/4\omega^2 a^2}{H_{n-2}^s - L_n^s + K_{n+2}^s} .$$

We may now deduce C_{n-2}^s, C_{n-4}^s, . . . C_{n+2}^s, C_{n+4}^s, . . . by means of the formulæ

$$\frac{C_{n-2}^s}{C_n^s} = \frac{H_{n-2}^s}{x_{n-2}^s}, \quad \frac{C_{n-4}^s}{C_n^s} = \frac{H_{n-2}^s H_{n-4}^s}{x_{n-2}^s x_{n-4}^s}, \ \&c.,$$

$$\frac{C_{n+2}^s}{C_n^s} = \frac{K_{n+2}^s}{y_n^s}, \quad \frac{C_{n+4}^s}{C_n^s} = \frac{K_{n+2}^s K_{n+4}^s}{y_n^s y_{n+2}^s}, \ \&c.,$$

and therefore

$$\zeta = \frac{h\gamma_n^s}{4\omega^2 a^2} \frac{e^{i(\lambda t + s\phi)}}{H_{n-2}^s - L_n^s + K_{n+2}^s} \left[\ldots + \frac{H_{n-2}^s H_{n-4}^s}{x_{n-2}^s x_{n-4}^s} P_{n-4}^s \right.$$
$$\left. + \frac{H_{n-2}^s}{x_{n-2}^s} P_{n-2}^s + P_n^s + \frac{K_{n+2}^s}{y_n^s} P_{n+2}^s + \frac{K_{n+2}^s K_{n+4}^s}{y_n^s y_{n+2}^s} P_{n+4}^s + \ldots \right]$$

It should be noticed that the term involving P_n^s need not here be the predominating term of the series within the square brackets; it will however be so when the value of λ for the disturbing force is in the neighbourhood of those roots of the period-equation which approximate to roots of $L_n^s = 0$. But if λ have as its value another root of the period-equation, say, for example, one which approximates to a root of $L_m^s = 0$, the series within square brackets will differ only by a constant multiplier from the series in the expression for the tide-height for the corresponding type of free oscillation, since the equations which determine the ratios of the C's are evidently the same in both cases. Hence, for values of λ in the neighbourhood of this one, the predominating term will be that which involves P_m^s. Consequently, when n and m differ widely, the numerical computation of these series will become laborious; for as we proceed away from the term containing P_n^s towards that involving P_m^s, the terms will at first increase in magnitude, and the convergence of the series will not assert itself until the term depending on P_m^s has been passed. These circumstances will not however occur in any of the cases of more practical interest.

Whatever be the nature of the disturbing potential it will be possible to expand its surface-value in a series of surface-harmonics. Thus the most general value of υ which can occur may be expressed in the form

$$\Sigma_\lambda \sum_{s=0}^{s=\infty} \sum_{n=s}^{n=\infty} \left[\gamma_n^s e^{i(\lambda t + s\phi)} + \delta_n^s e^{-i(\lambda t + s\phi)} \right] P_n^s(\mu).$$

The deformation of the surface due to each term may be calculated independently and the results superposed, so that the deformation at time t resulting from this disturbing force will be given by

$$\zeta = \frac{h}{4\omega^2 a^2} \Sigma_\lambda \sum_{s=0}^{s=\infty} \sum_{n=s}^{n=\infty} \left[\frac{\gamma_n^s e^{i(\lambda t + s\phi)} + \delta_n^s e^{-i(\lambda t + s\phi)}}{\mathrm{H}_{n-2}^s - \mathrm{L}_n^s + \mathrm{K}_{n+2}^s} \left\{ \ldots + \frac{\mathrm{H}_{n-2}^s}{x_{n-2}^s} \mathrm{P}_{n-2}^s + \mathrm{P}_n^s + \frac{\mathrm{K}_{n+2}^s}{y_n^s} \mathrm{P}_{n+2}^s + \ldots \right\} \right].$$

The corresponding formulæ when the depth is variable may be obtained by replacing h, γ_n^s, δ_n^s, H_n^s, K_n^s, L_n^s, x_n^s, y_n^s, by κ, G_n^s, Δ_n^s, $\mathfrak{H}_n^s$, $\mathfrak{K}_n^s$, $\mathfrak{L}_n^s$, ξ_n^s, η_n^s respectively, where κ, G_n^s, $\mathfrak{L}_n^s$, ξ_n^s, η_n^s are defined by the equations (35), (41), (42), and

$$\mathfrak{H}_n^s = \frac{\xi_n^s \eta_n^s}{\mathfrak{L}_n^s} - \frac{\xi_{n-2}^s \eta_{n-2}^s}{\mathfrak{L}_{n-2}^s} - \ldots$$

$$\mathfrak{K}_n^s = \frac{\xi_{n-2}^s \eta_{n-2}^s}{\mathfrak{L}_n^s} - \frac{\xi_n^s \eta_n^s}{\mathfrak{L}_{n+2}^s} - \ldots$$

while Δ_n^s is obtained by writing δ_n^s in place of γ_n^s in the right-hand member of (42).

§ 13. *Classification of Tides.*

In the last section we have reduced the problem of the evaluation of the forced tides due to any disturbing force to that of the development of the disturbing potential as a series of surface-harmonics. This development for the case of the disturbance of the ocean due to the attraction of the sun and moon has been already dealt with, and reference may be made to Professor DARWIN's article in the 'Encyclopædia Britannica' for a full account of it. We give here a short summary of the principal results of which we propose to make use.

The principal part of the disturbing potential will consist of spherical harmonic functions of the second order, and when expressed by means of zonal and tesseral harmonics the terms which occur will be of three types, characterized by the rank of the harmonic involved.

For the first type $s = 0$, and the corresponding tides will be expressible as series of zonal harmonics. For these types the value of λ will be small in comparison with that of ω, so that the period of the disturbance is long compared with a sidereal day. The terms will cease to be oscillatory when the orbital motion of the disturbing body is neglected. The tides generated by these parts of the disturbing potential have been already dealt with in Part I.

The terms of the second type are those for which $s = 1$; in certain of these terms the value of λ will be equal to ω, so that the period is rigorously equal to a sidereal day, while in the rest the period will reduce to a sidereal day when the orbital motion of the disturbing body is neglected. If n denote the mean orbital motion of the luminary, the "speeds" of the principal diurnal tides will be ω and $\omega - 2n$. We propose to neglect the sun's orbital motion, so that for each of the principal solar diurnal constituents we shall suppose that $\lambda = \omega$ rigorously. The same analysis will then apply to one of the lunar diurnal constituents, while in order to illustrate the effect of the departure of the period from exact coincidence with a sidereal day we shall evaluate independently the lunar diurnal constituent for which

$$\lambda\,(= \omega - 2n) = 0{\cdot}92700\,\omega.$$

The principal part of the tidal oscillations will be due to the third part of the disturbing potential, which involves harmonics of rank 2. The period for the tides due to these terms will differ but slightly from half a sidereal day, and will reduce to half a day exactly when the orbital motion of the disturbing body is neglected. We shall therefore assume that $\lambda = 2\omega$ rigorously for the solar semi-diurnal tides, while we shall take the value

$$\lambda/2\omega = 1 - n/\omega = 0{\cdot}96350$$

as typical of the lunar semi-diurnal constituents. The analysis applied to the solar constituents will be rigorously applicable to the sidereal luni-solar semi-diurnal tide usually denoted by the symbol K_2.

§ 14. *Special Cases.*

Instead of making use of the equation (31) as we have done in § 12, we may of course compute the forced tides by means of the equations (49), (50) of § 6, determining incidentally the constants D_n^s. Thus, if we suppose that all the γ's are zero except γ_{s+1}^s, we have the following equations for the determination of D_s^s, C_{s+1}^s, D_{s+2}^s, &c.

$$-\,N_s^s D_s^s + \frac{s^2(2s+1)}{2s+3}C_{s+1}^s = 0$$

$$\frac{(s+2)^2 \,.\, 1}{2s+1}D_s^s - M_{s+1}^s C_{s+1}^s + \frac{(s+1)^2(2s+2)}{2s+5}D_{s+2}^s = (s+1)^2(s+2)^2\frac{h\gamma_{s+1}^s}{4\omega^2 a^2}$$

$$\frac{(s+3)^2 \,.\, 2}{2s+3}C_{s+1}^s - N_{s+2}^s D_{s+2}^s + \frac{(s+2)^2(2s+3)}{2s+7}C_{s+3}^s = 0$$

$$\cdots\cdots\cdots\cdots$$

where the terms on the right are all zero, except in the second equation. Now it is evident that if $N_s^s = 0$, or

$$\lambda = \frac{2\omega}{s+1}$$

all these equations will be satisfied if

$$D_s^s = (s+1)^2(2s+1)\frac{h\gamma_{s+1}^s}{4\omega^2 a^2}$$

and

$$C_{s+1}^s = D_{s+2}^s = C_{s+3}^s = \ldots = 0.$$

It follows that if the disturbing potential be of order $s+1$ and rank s, and the period be $\frac{1}{2}(s+1)$ days, the tide will involve no rise and fall at the free surface, but will consist merely of horizontal currents. If we put $s=1$ the requisite period will be rigorously equal to a sidereal day, and the circumstances will correspond with those we have assumed to characterize the solar diurnal tides. We therefore conclude that in an ocean of uniform depth the solar diurnal tides will involve no rise and fall. We shall however see hereafter that for certain of the lunar diurnal tides the difference between the period and a sidereal day may be sufficiently great to render the rise and fall of considerable importance, unless the depth is very small.

We have seen in § 4 that the formulæ applicable to the case of variable depth may be deduced from those applicable to the case of uniform depth by replacing h by κ and C^s_n by $C^s_n + \frac{l}{4\omega^2a^2}\left[n(n+1)+\frac{2\omega s}{\lambda}\right]\Gamma^s_n$.

But if $\gamma^s_n = 0$, $\Gamma^s_n = -g_n C^s_n$, and therefore when $\lambda = 2\omega/(s+1)$ and all the γ's are zero except γ^s_{s+1}, C^s_{s+3}, C^s_{s+5}, &c., will all be zero, while

$$C^s_{s+1} + \frac{l}{4\omega^2a^2}\{(s+1)(s+2)+s(s+1)\}\Gamma^s_{s+1} = 0;$$

whence we obtain

$$C^s_{s+1} = -\frac{2(s+1)^2\frac{l\gamma^s_{s+1}}{4\omega^2a^2}}{1-2(s+1)^2\frac{lg_{s+1}}{4\omega^2a^2}}.$$

Thus the forced tide will be similar in type to the disturbing potential which produces it, though it will be inverted unless $l<0$ or $> \frac{2\omega^2a^2}{(s+1)^2 g_{s+1}}$.

This theorem will admit of application to the solar diurnal tides on putting $s=1$, in which case it reduces to a theorem given by LAPLACE.* The critical value of l, for a system comparable with the earth, is considerably greater than such depths as occur on the earth; hence, for depths comparable with that of the ocean, the diurnal tides will be inverted when the ocean is deeper at the equator than at the poles: they will however be direct when l is negative, so that the ocean is deeper at the poles than at the equator.

It is evident that the equations typified by (31) will all be satisfied with the C's all zero if $h=0$, since in this case the right-hand members will reduce to zero. In like manner the corresponding equations which apply to an ocean of variable depth will all be satisfied when $\kappa=0$, if for all values of n

$$C^s_n + \frac{l}{4\omega^2a^2}\left\{n(n+1)+\frac{2\omega s}{\lambda}\right\}\Gamma^s_n = 0.$$

* LAMB, 'Hydrodynamics,' § 212.

This equation leads to

$$C_n^s = -\frac{\frac{l\gamma_n^s}{4\omega^2a^2}\left\{n(n+1)+\frac{2\omega s}{\lambda}\right\}}{1-\frac{lg_n}{4\omega^2a^2}\left\{n(n+1)+\frac{2\omega s}{\lambda}\right\}}.$$

There exists then a certain law of depth, depending on the period, for which the tide will always be similar in type to the disturbing potential which produces it. This law of depth is expressed by the formula

$$h = l\left(\frac{\lambda^2}{4\omega^2} - \mu^2\right).$$

If we suppose that $\lambda = 2\omega$ rigorously, it reduces to

$$h = l(1-\mu^2),$$

so that the depth will be a maximum at the equator, and will gradually decrease on passing away from the equator to zero at the poles.*

For other values of λ the formula for h will make the depth negative at some parts of the surface unless l is positive and $\lambda > 2\omega$. The latter condition does not occur with any of the leading tidal constituents,† but it would hold good in the case of the semi-diurnal tides due to a satellite whose motion in its orbit was retrograde. If however we neglect the mutual attraction of the waters, the theorem under discussion may be supposed to apply to an ocean covering that part of the surface over which h is positive, the remaining parts of the surface being supposed to consist of continents. When l is positive, these continents must, for the lunar semi-diurnal tides, reduce to small circumpolar islands, while for the same tides when l is negative they will cover the whole globe with the exception of two small seas surrounding the poles.

For the diurnal-tides, the shores must coincide with parallels of latitude approximately 30° north and south of the equator, while for the tides of long period the appropriate forms of sea will be bounded by two parallels nearly coincident with the equator. A change in the sign of l in all cases involves an interchange between the seas and the land.

It should be noticed that the formulæ (12) make U, V infinite at the points where $\mu = \pm\frac{\lambda}{2\omega}$, that is, at the shores. This indicates that the neglect of the squares of the velocities is not allowable in the neighbourhood of the coasts no matter how small the amplitude of vibration may be, and seems to point to the existence of "breakers" as an essential accompaniment of the tides.

* *Cf.* LAMB, 'Hydrodynamics,' § 213.

† There will be small tidal constituents depending on higher powers of the moon's parallax for which λ exceeds 2ω. *Cf.* DARWIN, "Harmonic Analysis of Tidal Observations." 'Brit. Assoc. Report,' 1883 (Southport), § 3.

§ 15. *Solar Semi-diurnal Tides.*

In the last section we have considered some special cases in which the tide-height is expressible by a single term. In general it will however only be expressible by a series of terms. It may be shown, as in § 5 of Part I., that this series will be finite when the law of depth is such that

$$1 - \frac{lg_n}{4\omega^2 a^2}\left\{n(n+1) + \frac{2\omega s}{\lambda}\right\} = 0,\text{*}$$

where n is an integer, $n - s$ being even or odd according as we are dealing with the symmetrical or asymmetrical types of rank s. The values of l determined from this equation will in general involve λ, but for large values of n, they will approximate to the same values of l as those required for the expression in finite terms of the long-period tides, since for such values $2\omega s/\lambda$ will be small compared with $n(n+1)$.

In other cases the expression for the tide-height will involve infinite series. We deal in the present section with the case where l is zero, so that the depth is uniform.

The numerical computation of the semi-diurnal tides admits of special simplicity when $\lambda = 2\omega$ exactly. Putting $\lambda = 2\omega$, $s = 2$, in the formulæ (30), (29) we obtain

$$x_n^2 = \frac{n-1}{(n+3)(2n+1)(2n+3)}, \qquad y_n^2 = \frac{n+4}{n(2n+3)(2n+5)},$$

$$\Lambda_n^2 = \frac{(n-1)(n+2)}{n^2(n+1)^2} - \frac{(n-1)^2(n+2)}{n^2(n+1)(2n-1)(2n+1)} - \frac{(n+2)^2(n-1)}{n(n+1)^2(2n+1)(2n+3)},$$

the last of which gives on reduction

$$\Lambda_n^2 = \frac{2(n-1)(n+2)}{n(n+1)(2n-1)(2n+3)}.$$

This general formula for Λ_n^2 fails to hold when $n = 2$; for, in this case, $n - s$ and $n(n-1) - 2\omega s/\lambda$ are both zero, and therefore the second fraction involved in the expression for Λ_n^2 is indeterminate. To determine its limiting form we must first suppose the period slightly different from half a day, so that λ is not rigorously equal to 2ω; the formula (29) then gives

$$\Lambda_2^2 = \frac{\lambda^2}{4\omega^2}\,\frac{2.3 - 4\omega/\lambda}{2^2.3^2} - \frac{4^2.1.5}{3^2.5.7\{3.4 - 4\omega/\lambda\}},$$

which, on putting $\lambda = 2\omega$, reduces to

$$\Lambda_2^2 = \frac{4}{2^2.3^2} - \frac{1.4^2}{3^2.7.10} = \frac{3}{5.7}.$$

The formulæ for x_n^2, y_n^2, Λ_n^2 are now in a convenient form for logarithmic computation, and we may readily deduce the following numerical values

* *Cf.* LAPLACE, 'Méc. Cél.,' Part I., Book IV., § 7.

$\Lambda_2^2 = 0{\cdot}085714$ $\quad \Lambda_{10}^2 = 0{\cdot}004494$

$\Lambda_4^2 = 0{\cdot}0233766$ $\quad \Lambda_{12}^2 = 0{\cdot}003179$

$\Lambda_6^2 = 0{\cdot}011544$ $\quad \Lambda_{14}^2 = 0{\cdot}002367$

$\Lambda_8^2 = 0{\cdot}006823$ $\quad \Lambda_{16}^2 = 0{\cdot}001830$

$\log x_2^2 = 3{\cdot}75696$ $\quad \log x_{10}^2 = \bar{3}{\cdot}1564$

$\log x_4^2 = \bar{3}{\cdot}63639$ $\quad \log x_{12}^2 = \bar{3}{\cdot}0360$

$\log x_6^2 = \bar{3}{\cdot}45469$ $\quad \log x_{14}^2 = \bar{4}{\cdot}9297$

$\log x_8^2 = \bar{3}{\cdot}29451$

$\log y_2^2 = \bar{2}{\cdot}67778$ $\quad \log y_{10}^2 = \bar{3}{\cdot}3865$

$\log y_4^2 = \bar{2}{\cdot}14569$ $\quad \log y_{12}^2 = 3{\cdot}2312$

$\log y_6^2 = \bar{3}{\cdot}81531$ $\quad \log y_{14}^2 = \bar{3}{\cdot}0993$

$\log y_8^2 = \bar{3}{\cdot}57512$

Taking $hg/4\omega^2a^2 = \frac{1}{40}$, and $\rho/\sigma_0 = 0{\cdot}18093$, the formula (32) leads to

$L_2^2 = +\ 0{\cdot}063428$ $\quad L_{10}^2 = -\ 0{\cdot}019860$

$L_4^2 = -\ 0{\cdot}0001156$ $\quad L_{12}^2 = -\ 0{\cdot}02128$

$L_6^2 = -\ 0{\cdot}012412$ $\quad L_{14}^2 = -\ 0{\cdot}02216$

$L_8^2 = -\ 0{\cdot}017379$ $\quad L_{16}^2 = -\ 0{\cdot}0228$

Thus if we neglect K_{18}^2, and make use of the formula

$$K_n^s = \frac{x_{n-2}^s y_{n-2}^s}{L_n^s - K_{n+2}^s},$$

we obtain in succession

$\log K_{16}^2 = n\,\bar{5}{\cdot}671$ $\quad \log K_8^2 = n\,\bar{3}{\cdot}03947$

$\log K_{14}^2 = n\,\bar{5}{\cdot}9226$ $\quad \log K_6^2 = n\,\bar{3}{\cdot}72835$

$\log K_{12}^2 = n\,\bar{4}{\cdot}2165$ $\quad \log K_4^2 = \bar{2}{\cdot}71588$

$\log K_{10}^2 = n\,\bar{4}{\cdot}5753$

and therefore, since $C_{n+2}^s/C_n^s = K_{n+2}^s/y_n^s$, we find

$\log (C_4^2/C_2^2) = 0{\cdot}03810$ $\quad \log (C_{12}^2/C_{10}^2) = n\,\bar{2}{\cdot}8300$

$\log (C_6^2/C_4^2) = n\,\bar{1}{\cdot}58266$ $\quad \log (C_{14}^2/C_{12}^2) = n\,\bar{2}{\cdot}6914$

$\log (C_8^2/C_6^2) = n\,\bar{1}{\cdot}22417$ $\quad \log (C_{16}^2/C_{14}^2) = n\,2{\cdot}572$

$\log (C_{10}^2/C_8^2) = n\,\bar{1}{\cdot}0002$

But if $\mathfrak{C}_n^s P_n^s(\mu) e^{i(\lambda t + s\phi)}$ denote the height of the 'equilibrium' tide resulting from a disturbing potential $\gamma_n^s P_n^s(\mu) e^{i(\lambda t + s\phi)}$, we have

$$\gamma_n^s = g_n \mathfrak{C}_n^s.$$

Therefore

$$C_2^2 = -\frac{h\gamma_2^2/4\omega^2 a^2}{L_2^2 - K_4^2} = -\frac{hg_2/4\omega^2 a^2}{L_2^2 - K_4^2}\mathfrak{C}_2^2$$

and on introducing the numerical values for h, g_2, L_2^2, K_4^2 we obtain

$$C_2^2 = -1\cdot9476\ \mathfrak{C}_2^2.$$

Thus we have :

$$\log(C_4^2/\mathfrak{C}_2^2) = n\,0\cdot32760 \qquad \log(C_{12}^2/\mathfrak{C}_2^2) = n\,\bar{4}\cdot9646$$
$$\log(C_6^2/\mathfrak{C}_2^2) = \bar{1}\cdot91025 \qquad \log(C_{14}^2/\mathfrak{C}_2^2) = \bar{5}\cdot6560$$
$$\log(C_8^2/\mathfrak{C}_2^2) = n\,\bar{1}\cdot13442 \qquad \log(C_{16}^2/\mathfrak{C}_2^2) = n\,\bar{6}\cdot228\ ;$$
$$\log(C_{10}^2/\mathfrak{C}_2^2) = \bar{2}\cdot1346$$

and therefore if we suppose the exponential or trigonometrical factor to be involved in $\mathfrak{C}_2^2$, so that the height of the equilibrium tide is expressed by $\mathfrak{C}_2^2 P_2^2(\mu)$, the height of the corresponding dynamical tide is given by

$$\zeta = \mathfrak{C}_2^2[-1\cdot9476\,P_2^2 - 2\cdot12617\,P_4^2 + 0\cdot81331\,P_6^2 - 0\cdot13628\,P_8^2$$
$$+ 0\cdot01363\,P_{10}^2 - 0\cdot000922\,P_{12}^2 + 0\cdot000045\,P_{14}^2 - 0\cdot000002\,P_{16}^2 + \ldots].$$

For points lying on the equator we have $\mu = 0$, and it may be shown that in this case

$$P_{2n}^2 = (-)^{n+1}\frac{3.5\ldots(2n+1)}{2.4\ldots(2n-2)},$$

whence we deduce

$$\log P_2^2 = 0\cdot47712 \qquad \log P_{10}^2 = 1\cdot4325$$
$$\log P_4^2 = n\,0\cdot87506 \qquad \log P_{12}^2 = n\,1\cdot5464$$
$$\log P_6^2 = 1\cdot11810 \qquad \log P_{14}^2 = 1\cdot643$$
$$\log P_8^2 = n\,1\cdot29419 \qquad \log P_{16}^2 = n\,1\cdot73.$$

Thus the values of the successive terms of the series within the square brackets for points at the equator are

$$-5\cdot8428 + 15\cdot9463 + 10\cdot6746 + 2\cdot6829 + 0\cdot3690 + 0\cdot0324 + 0\cdot0020 + 0\cdot0001,$$

which, on addition, give 23·8645. But the height of the corresponding equilibrium-tide at the equator is $3\mathfrak{C}_2^2$, and therefore the ratio of the height of the tide to that of the corresponding equilibrium tide at the equator is

$$+ 7\cdot9548.$$

In like manner the tide-height in any other latitude may be compared with the equilibrium tide-height, but the process will be laborious in the absence of tables of the functions P_n^s.*

The above example has been treated in some detail as illustrative of the method to be employed for the computation of the forced tides by infinite series; the chief part of the labour is involved in the determination of the quantities x_n^s, y_n^s, Λ_n^s, but when once these have been determined, since they do not involve the depth, it is easy without much additional labour to multiply cases for different depths. Besides the case already considered, which corresponds to a depth of about 7260 feet, I have computed the series for depths of 14,520, 29,040, and 58,080 feet, corresponding with the values $\frac{1}{20}$, $\frac{1}{10}$, and $\frac{1}{5}$ for $hg/4\omega^2a^2$. For these depths the series within the square brackets is replaced by

$$- 0\cdot83227 P_2^2 + 0\cdot21694 P_4^2 - 0\cdot02615 P_6^2 + 0\cdot00180 P_8^2 - 0\cdot000080 P_{10}^2 + 0\cdot000003 P_{12}^2 + \ldots,$$

$$- 191\cdot925 P_2^2 + 15\cdot696 P_4^2 - 0\cdot8082 P_6^2 + 0\cdot0256 P_8^2 - 0\cdot0005 P_{10}^2 + \ldots$$

and

$$1\cdot9610 P_2^2 - 0\cdot06823 P_4^2 + 0\cdot00164 P_6^2 - 0\cdot000025 P_8^2 + \ldots$$

respectively, giving for the ratio of the tide-height to the equilibrium tide-height at the equator the values

$$- 1\cdot5016, \qquad - 234\cdot87, \qquad + 2\cdot1389.$$

When the depth of the ocean is greater than 58,080 feet the tides are therefore direct at the equator. They gradually increase in magnitude as the depth decreases, and become infinite and change sign for some critical value of the depth rather in excess of 29,040 feet after which, for further decrease of the depth, they remain inverted until a second critical value is reached which is somewhat greater than 7260 feet, when a second change of sign occurs. The very large coefficients which appear when $hg/4\omega^2a^2 = \frac{1}{10}$ indicate that for this depth there is a period of free oscillation of semi-diurnal type whose period differs but slightly from half-a-day. On reference to the tables of § 9 it will be seen that we have, in fact, evaluated this period as 12 hours 1 minute, while for the case $hg/4\omega^2a^2 = \frac{1}{40}$ we have found a period of 12 hours 5 minutes. We see then that though, when the period of the forced oscillation differs from that of one of the types of free oscillation by as little as one minute, the forced tide may be nearly 250 times as great as the corresponding equilibrium tide, a

* The zonal harmonics from P_1 up to P_7 have been tabulated by GLAISHER, 'Brit. Assoc. Reports,' 1879, but I do not know of the existence of any Tables of the Tesseral Harmonics, with the exception of a few given by THOMSON and TAIT, 'Nat. Phil.,' vol. 2, § 784.

difference of 5 minutes between these periods will be sufficient to reduce the tide to less than ten times the corresponding equilibrium tide. It seems then that the tides will not tend to become excessively large unless there is very close agreement with the period of one of the free oscillations.

The critical depths for which the forced tides here treated of become infinite are those for which a period of free oscillation coincides exactly with 12 hours. They may be ascertained by putting $\lambda = 2\omega$ in the period-equation for the free oscillations and treating this equation as an equation for the determination of h. The roots may be found by trial and error as in § 9, the approximate values with which to commence the trials being suggested by the discussion already given. The two largest roots are found to be given by

$$hg/4\omega^2a^2 = 0\cdot10049, \qquad hg/4\omega^2a^2 = 0\cdot02545,$$

and the corresponding critical depths are about 29,182 feet and 7375 feet.

We have hitherto supposed that $\rho/\sigma_0 = 0\cdot18093$, but for purposes of comparison I have also examined the case where $\rho/\sigma_0 = 0$, that is where the mutual attraction of the waters is neglected. The series for ζ in this case become

$$\zeta = \mathfrak{C}_2^2[1\cdot0927\,P_2^2 + 1\cdot91817\,P_4^2 - 0\cdot66909\,P_6^2 + 0\cdot10701\,P_8^2 - 0\cdot01036\,P_{10}^2 + 0\cdot000683\,P_{12}^2 - 0\cdot000033\,P_{14}^2 + 0\cdot000001\,P_{16}^2 - \ldots]$$

$$\zeta = \mathfrak{C}_2^2[-1\cdot0733\,P_2^2 + 0\cdot24502\,P_4^2 - 0\cdot02790\,P_6^2 + 0\cdot00185\,P_8^2 - 0\cdot000080\,P_{10}^2 + 0\cdot000002\,P_{12}^2 - \ldots]$$

$$\zeta = \mathfrak{C}_2^2[9\cdot34370\,P_2^2 - 0\cdot70311\,P_4^2 + 0\cdot03449\,P_6^2 - 0\cdot00106\,P_8^2 + 0\cdot000022\,P_{10}^2 - \ldots]$$

$$\zeta = \mathfrak{C}_2^2[1\cdot7739\,P_2^2 - 0\cdot05750\,P_4^2 + 0\cdot00132\,P_6^2 - 0\cdot00020\,P_8^2 + \ldots]$$

or the depths 7260, 14,520, 29,040, 58,080 feet respectively. From these series we deduce as the ratio of the tide to the equilibrium tide at the equator the four values

$$-7\cdot4343, \qquad -1\cdot8208, \qquad +11\cdot2595, \qquad +1\cdot9236,$$

results which agree, except in the third case, with the numbers given by Professor LAMB* deduced from the numerical formulæ of LAPLACE.

It will be seen that in three cases out of the four here considered the effect of the mutual gravitation of the waters is to increase the ratio of the tide to the equilibrium tide. In two of the cases the sign is also reversed. This of course results from the fact that, whereas when $\rho/\sigma_0 = 0\cdot18093$ one of the periods of free oscillation is

* By a careful re-computation of the semi-diurnal tide for the case $\beta = 10$ (notation of Professor LAMB) I find the following series more accurate than that given for ζ/H''' :—

$$\nu^2 + 6\cdot1915\,\nu^4 + 3\cdot2447\,\nu^6 + 0\cdot7234\,\nu^8 + 0\cdot0919\,\nu^{10} + 0\cdot0076\,\nu^{12} + 0\cdot0004\,\nu^{14} + \ldots$$

This series reduces to 11·2595 when $\nu = 1$, thus agreeing with the result obtained above.

rather greater than 12 hours, when $\rho/\sigma_0 = 0$ the corresponding period will be less than 12 hours

§ 16. *Lunar Semi-diurnal Tides.*

A similar method to that of the last section may be used to evaluate the lunar semi-diurnal tides for which we take $\lambda/2\omega = 0{\cdot}96350$. The arithmetical work is, however, more severe, in consequence of the fact that the quantities x_n^s, y_n^s, Λ_n^s must be evaluated from the formulæ (29), (30) which do not assume the simple forms obtained in the last section. Substituting in these formulæ the value of $\lambda/2\omega$ quoted above we deduce

$$\Lambda_2^2 = 0{\cdot}075603 \qquad \Lambda_{10}^2 = 0{\cdot}003845$$
$$\Lambda_4^2 = 0{\cdot}019864 \qquad \Lambda_{12}^2 = 0{\cdot}002750$$
$$\Lambda_6^2 = 0{\cdot}009856 \qquad \Lambda_{14}^2 = 0{\cdot}002028$$
$$\Lambda_8^2 = 0{\cdot}005834 \qquad \Lambda_{16}^2 = 0{\cdot}001968\,;$$

$$\log x_2^2 = \bar{3}{\cdot}76026 \qquad \log x_{10}^2 = \bar{3}{\cdot}1566$$
$$\log x_4^2 = \bar{3}{\cdot}63756 \qquad \log x_{12}^2 = \bar{3}{\cdot}0362$$
$$\log x_6^2 = \bar{3}{\cdot}45530 \qquad \log x_{14}^2 = \bar{4}{\cdot}9299\,;$$
$$\log x_8^2 = \bar{3}{\cdot}29488$$

$$\log y_2^2 = \bar{2}{\cdot}68108 \qquad \log y_{10}^2 = \bar{3}{\cdot}3867$$
$$\log y_4^2 = \bar{2}{\cdot}14687 \qquad \log y_{12}^2 = \bar{3}{\cdot}2313$$
$$\log y_6^2 = \bar{3}{\cdot}81592 \qquad \log y_{14}^2 = \bar{3}{\cdot}0994.$$
$$\log y_8^2 = \bar{3}{\cdot}57549$$

Our procedure is now exactly similar to that of the last section. Thus, if the height of the equilibrium tide be

$$\mathfrak{C}_2^2\, P_2^2(\mu),$$

we find, when $hg/4\omega^2a^2 = \frac{1}{40}$ and $\rho/\sigma_0 = 0{\cdot}18093$,

$$\zeta = \mathfrak{C}_2^2[0{\cdot}10396\,P_2^2 + 0{\cdot}57998\,P_4^2 - 0{\cdot}19273\,P_6^2 + 0{\cdot}03054\,P_8^2 - 0{\cdot}002960\,P_{10}^2 + 0{\cdot}000196\,P_{12}^2 - 0{\cdot}000010\,P_{14}^2 + \ldots].$$

Similarly, when $hg/4\omega^2a^2 = \frac{1}{20}$,

$$\zeta = \mathfrak{C}_2^2[-1{\cdot}0647\,P_2^2 + 0{\cdot}24038\,P_4^2 - 0{\cdot}02774\,P_6^2 + 0{\cdot}001867\,P_8^2 - 0{\cdot}000082\,P_{10}^2 + 0{\cdot}000003\,P_{12}^2 - \ldots]\,;$$

when $hg/4\omega^2a^2 = \frac{1}{10}$,

$$\zeta = \mathfrak{C}_2^2[9{\cdot}1181\,P_2^2 - 0{\cdot}71533\,P_4^2 + 0{\cdot}03621\,P_6^2 - 0{\cdot}001136\,P_8^2 + 0{\cdot}000024\,P_{10}^2 - \ldots]\,;$$

and when $hg/4\omega^2a^2 = \frac{1}{5}$,

$$\zeta = \mathfrak{C}_2^2 [1\cdot7646\, P_2^2 - 0\cdot06057\, P_4^2 + 0\cdot001447\, P_6^2 - 0\cdot000022\, P_8^2 + \ldots].$$

From these series we find for the ratio of the tide-heights to the equilibrium tide-heights at the equator the four values

$$-2\cdot4187, \quad -1\cdot8000, \quad +11\cdot0725, \quad +1\cdot9225.$$

On comparison of these numbers with those obtained for the solar tides in the preceding section, we see that for a depth of 7260 feet the solar tides will be direct while the lunar tides will be inverted, the opposite being the case when the depth is 29,040 feet. This is, of course, due to the fact that in each of these cases there is a period of free oscillation intermediate between twelve solar (or, more strictly, sidereal) hours and twelve lunar hours. The critical depths for which the lunar tides become infinite are found to be 26,044 feet and 6448 feet.

Consequently this phenomenon will occur if the depth of the ocean be between 29,182 feet and 26,044 feet, or between 7375 feet and 6448 feet. An important consequence would be that for depths lying between these limits the usual phenomena of spring and neap tides would be reversed, the higher tides occurring when the moon is in quadrature, and the lower at new and full moon.*

There appears then to be a considerable range of depth comparable with the mean depth of the ocean over which the reversal of the spring and neap tide phenomena would take place, but in that the actual tides are highest in the neighbourhood of new and full moon we conclude that the effective depth of the ocean does not lie within this range, and that none of the periods of free oscillation of the actual ocean lie between twelve solar hours and twelve lunar hours. The true effective depth is almost certainly less than 26,044 feet, and therefore both solar and lunar tides will be in the main inverted, though the configuration of the land and of the ocean bed will probably give rise to considerable variations of phase in different places.

The shortest period of free oscillation of the second class for the case $s = 2$ approximates to, but is in excess of, three days. But if n denotes the moon's mean orbital motion, the speed of the lunar semi-diurnal tide is

$$2\,(\omega - n).$$

If we equate this to $\frac{1}{3}\,\omega$, we obtain

$$n = \tfrac{5}{6}\,\omega.$$

Hence, if the moon's orbital motion were accelerated, or the earth's rotation retarded, until the month and day were in a ratio less than 6 : 5, it would be possible for the period of the lunar semi-diurnal tide to confound itself with one of the periods of the oscillations of the second class, and the tides would then tend to become very large.

* *Cf.* KELVIN, 'Popular Lectures,' vol. 2, p. 22 (footnote).

§ 17. *Solar Semi-diurnal Tides in an Ocean of Variable Depth.*

To evaluate the tides when the depth is a function of the latitude we must make use of the formulæ of § 4. The method will be sufficiently illustrated by the computation of the solar semi-diurnal tide for the case where

$$hg/4\omega^2a^2 = \tfrac{1}{20} + \tfrac{1}{30}\sin^2\theta,$$

or, where

$$h = (14{,}520 + 9680\sin^2\theta) \text{ feet},$$

θ denoting the co-latitude.

Putting $lg/4\omega^2a^2 = \frac{1}{30}$, and making use of the numerical values found in § 15 for x_n^2, y_n^2, Λ_n^2, we obtain from (41)

$$\log \xi_2^2 = \bar{3}{\cdot}63907 \qquad \log \xi_6^2 = n\,\bar{3}{\cdot}06261$$
$$\log \xi_4^2 = \bar{3}{\cdot}12901 \qquad \log \xi_8^2 = n\,\bar{3}{\cdot}4369$$

$$\log \eta_2^2 = \bar{2}{\cdot}17040 \qquad \log \eta_6^2 = n\,\bar{3}{\cdot}95767$$
$$\log \eta_4^2 = n\,\bar{3}{\cdot}75361 \qquad \log \eta_8^2 = n\,\bar{3}{\cdot}9962$$

while, when $\dfrac{\kappa g}{4\omega^2a^2} = \frac{1}{20}$, we obtain

$$\mathfrak{L}_2^2 = +\,0{\cdot}020765 \qquad \mathfrak{L}_8^2 = -\,0{\cdot}05787$$
$$\mathfrak{L}_4^2 = -\,0{\cdot}039717 \qquad \mathfrak{L}_{10}^2 = -\,0{\cdot}0606$$
$$\mathfrak{L}_6^2 = -\,0{\cdot}052592.$$

From these we deduce in succession, on neglecting $\mathfrak{K}_{12}^2$,

$$\log \mathfrak{K}_{10}^2 = n\,\bar{4}{\cdot}651 \qquad \log \mathfrak{K}_6^2 = \bar{4}{\cdot}1632$$
$$\log \mathfrak{K}_8^2 = n\,\bar{4}{\cdot}2612 \qquad \log \mathfrak{K}_4^2 = \bar{3}{\cdot}20891$$

and

$$\log \mathfrak{H}_2^2 = \bar{3}{\cdot}49214.$$

But the first two of equations (40) give

$$-\,\mathfrak{L}_2^2 C_2^2 + \eta_2^2 C_4^2 = \left[\frac{\kappa}{4\omega^2a^2} + 8\,\frac{l\Lambda_2^2}{4\omega^2a^2}\right]\gamma_2^2 = \left[\frac{\kappa g_2}{4\omega^2a^2} + 8\,\frac{lg_2}{4\omega^2a^2}\,\Lambda_2^2\right]\mathfrak{C}_2^2,$$

$$\xi_2^2 C_2^2 - (\mathfrak{L}_4^2 - \mathfrak{K}_6^2)\,C_4^2 = -\,8\,\frac{l\gamma_2^2}{4\omega^2a^2}\,x_2^2 = -\,8x_2^2\,\frac{lg_2}{4\omega^2a^2}\,\mathfrak{C}_2^2,$$

which, on solution, yield

$$C_2^2 = -\left[\frac{\kappa g_2}{4\omega^2a^2} + 8\,\frac{lg_2}{4\omega^2a^2}\,\Lambda_2^2\right]\frac{\mathfrak{C}_2^2}{\mathfrak{L}_2^2 - \mathfrak{K}_4^2} + 8x_2^2\,\frac{lg_2}{4\omega^2a^2}\,\frac{\mathfrak{H}_2^2}{\xi_2^2}\,\frac{\mathfrak{C}_2^2}{\mathfrak{L}_4^2 - \mathfrak{H}_2^2 - \mathfrak{K}_6^2},$$

$$C_4^2 = -\left[\frac{\kappa g_2}{4\omega^2a^2} + 8\,\frac{lg_2}{4\omega^2a^2}\,\Lambda_2^2\right]\frac{\mathfrak{K}_4^2}{\eta_2^2}\,\frac{\mathfrak{C}_2^2}{\mathfrak{L}_2^2 - \mathfrak{K}_4^2} + 8x_2^2\,\frac{lg_2}{4\omega^2a^2}\,\frac{\mathfrak{C}_2^2}{\mathfrak{L}_4^2 - \mathfrak{H}_2^2 - \mathfrak{K}_6^2}.$$

On substituting the numerical values for the quantities on the right, we obtain

$$C_2^2 = -2\cdot9242\ \mathfrak{C}_2^2, \qquad C_4^2 = 0\cdot28546\ \mathfrak{C}_2^2.$$

The remaining constants may now be computed from the formulæ $C_6^2/C_4^2 = \mathfrak{K}_6^2/\eta_4^2$, $C_8^2/C_6^2 = \mathfrak{K}_8^2/\eta_6^2$ &c., and we finally obtain

$$\zeta = \mathfrak{C}_2^2[-2\cdot9242P_2^2 + 0\cdot28546P_4^2 - 0\cdot00733P_6^2 - 0\cdot000147P_8^2 - 0\cdot000007P_{10}^2 - \ldots].$$

This makes the ratio of the height of the tide to that of the equilibrium tide at the equator

$$-3\cdot6690.$$

The tide will evidently be in the main inverted, the longest period of free oscillation of the first class being in excess of twelve hours.

As a further example, I have computed the series for ζ when the depth is given by the formula

$$\frac{hg}{4\omega^2a^2} = \tfrac{1}{10} - \tfrac{1}{30}\sin^2\theta,$$

that is, when the depth is 29,040 feet at the poles and shallows to 19,360 feet at the equator. This series is found to be

$$\zeta = \mathfrak{C}_2^2[-2\cdot3661P_2^2 + 0\cdot35649P_4^2 - 0\cdot03953P_6^2 + 0\cdot00376P_8^2 - 0\cdot000333P_{10}^2 + 0\cdot000029P_{12}^2 - 0\cdot000002P_{14}^2 + \ldots],$$

making the ratio of the tide to the equilibrium tide at the equator

$$-3\cdot4583.$$

If we put $\frac{lg}{4\omega^2a^2} = \frac{1}{30}$, and replace λ by 2ω in the period-equation, regarding this as an equation for κ, the largest root is found to be

$$\kappa = 21{,}765.$$

Thus there will be a period of free oscillation coinciding exactly with twelve hours when

$$h = (21{,}765 + 9680\sin^2\theta)\ \text{feet};$$

this formula makes the polar depth 21,765 feet, the equatorial depth 31,445 feet, and the mean depth 28,222 feet.

In like manner, when $\frac{lg}{4\omega^2a^2} = -\frac{1}{30}$, there will also be a period of free oscillation agreeing exactly with twelve hours, when the polar depth is 36,970 feet, the equatorial depth 26,290 feet, and the mean depth 29,517 feet.

§ 18. *Diurnal Tides.*

It has been shown in § 14 that the diurnal tidal constituents whose periods are equal to a sidereal day will involve no rise and fall at the free surface when the depth of the ocean is uniform. This theorem will be rigorously applicable to the luni-solar diurnal constituent usually designated by the initial K_1, while it may also be supposed to apply with a fair degree of accuracy to each of the solar diurnal constituents since the motion of the sun in his orbit is sufficiently slow. There will however be an important lunar diurnal constituent for which the speed is $0{\cdot}92700\omega$, in dealing with which we propose to take into account the difference between the period and a sidereal day. The method of computation is exactly similar to that used for the lunar semi-diurnal tides, and thus we find when $hg/4\omega^2a^2 = \frac{1}{40}$,

$$\zeta = \mathfrak{C}_2^1[-0{\cdot}07638P_2^1 + 0{\cdot}03543P_4^1 - 0{\cdot}00845P_6^1 + 0{\cdot}001207P_8^1 - 0{\cdot}000114P_{10}^1 + 0{\cdot}000008P_{12}^1 - \ldots];$$

when $hg/4\omega^2a^2 = \frac{1}{20}$,

$$\zeta = \mathfrak{C}_2^1[-0{\cdot}1691P_2^1 + 0{\cdot}04738P_4^1 - 0{\cdot}00628P_6^1 + 0{\cdot}000480P_8^1 - 0{\cdot}000024P_{10}^1 + 0{\cdot}000001P_{12}^1 - \ldots];$$

when $hg/4\omega^2a^2 = \frac{1}{10}$,

$$\zeta = \mathfrak{C}_2^1[-0{\cdot}4145P_2^1 + 0{\cdot}06576P_4^1 - 0{\cdot}00464P_6^1 + 0{\cdot}000184P_8^1 - 0{\cdot}000005P_{10}^1 + \ldots];$$

and when $hg/4\omega^2a^2 = \frac{1}{5}$,

$$\zeta = \mathfrak{C}_2^1[-1{\cdot}4428P_2^1 + 0{\cdot}1231P_4^1 - 0{\cdot}00449P_6^1 + 0{\cdot}00009P_8^1 - 0{\cdot}000001P_{10}^1 + \ldots].$$

It appears, then, that these tides will increase with the depth, and that they will be in the main inverted. For small depths the rise and fall will be small, but with a depth as great as 58,080 feet the tide will be in excess of the equilibrium-tide. The type will tend to approximate more and more closely to that represented by a second order harmonic alone as the depth increases.

So long as the depth is uniform the tidal constituents whose periods are rigorously equal to a sidereal day will never tend to become infinite, and consequently no period of free oscillation of the type of the diurnal tides can coincide exactly with one day. As ω diminishes however the shortest period of the second class, which for the depths under consideration is longer than the period of the lunar diurnal constituents, approximates to one day, and attains this as a limiting value when $\omega = 0$. Hence, as ω diminishes, or h increases, the largest root of the second class must pass through the value 0·92700, thus rendering one of the lunar diurnal constituents infinite. This accounts for the rapid increase in the coefficients in the series given above for these tides as h increases.

The roots of the first class must however all be greater than unity, no matter how great the depth may be. Since they all decrease with the depth, they must approach finite limiting values greater than unity as the depth diminishes to zero.

The tides of rigorously diurnal period will become infinite when the depth is variable if l assumes the value

$$\frac{1}{2}\frac{\omega^2 a^2}{g_2},$$

so that with this value of l we may anticipate that there will be a period of free oscillation exactly a sidereal day in duration. Now the above value of l will require that the surface of the solid earth should be rigorously spherical in order that the free surface of the ocean may be an equipotential surface under gravity and centrifugal force. It is easy to see why in this case a free oscillation of rigorously diurnal period must exist. For if the water be set in rotation as a solid body about an axis not rigorously coincident with the rotation-axis of the solid earth, and the form of the free surface be adjusted for equilibrium under centrifugal force about the new axis of rotation, there will be no forces acting which tend to modify this state of motion, and it will continue permanently, provided the system be free from friction. The motion of the water will be steady in space, but it will be oscillatory with a period of one day relatively to the solid earth.

It is easy to verify that in the forced oscillations of rigorously diurnal period the motion of the water is of like character, involving no relative motion of the parts and being steady in space. In this case the axis about which the rotation of the water takes place lies in the plane containing the earth's polar axis and the disturbing body.

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